Bayesian Persuasion with Multiple Receivers*

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February 28, 2015

Abstract

This paper investigates the role of persuasion mechanisms in collective decision-making. A biased sender adopts a Bayesian persuasion mechanism to provide a committee of uninformed receivers with signals about the unknown state of the world. We compare public persuasion with private persuasion. We find that the sender can always reach the concave closure of the set of possible expected payoffs under public persuasion, regardless of the number of generated signals. The sender is weakly worse off under private persuasion. We also provide conditions under which the receivers' welfare from private persuasion dominates that from public persuasion. Moreover, voting fully aggregates receivers' private information in the state where the sender and receivers' preferences are perfectly aligned, while full information aggregation may fail in other states.

Keywords: *strategic information transmission, sender-receiver game, Bayesian persuasion, voting*

JEL Classification: *C72; D72; D82; D83*

*I am indebted to Sourav Bhattacharya, Andreas Blume, John Duffy, Isa Hafalir, Luca Rigotti, and Roee Teper for their dedication, help, and guidance. I would also like to thank Jimmy Chan, Mehmet Ekmekci, Maria Goltsman, Johannes Hörner, Vijay Krishna, Ming Li, Tymofiy Mylovanov, Gregory Pavlov, Ina Taneva, Junjie Zhou and the seminar and conference participants at Carnegie Mellon University, University of Pittsburgh, Shanghai University of Finance and Economics, Peking University HSBC Business School, the Fall 2011 Midwest Economic Theory Meeting, the 38th Annual Conference of the Eastern Economic Association, the 76th Annual Conference of the Midwest Economic Association, the 23rd International Conference on Game Theory at Stony Brook, the 4th Game Theory World Congress, and the 47th Annual Conference of the Canadian Economic Association for helpful comments and suggestions. I also thank the National Science Foundation of China for funding support (Grant No. 71403228). All errors are my own.

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1 Introduction

In many social or economic situations, a group of decision–makers receives advice from an informed agent. The agent designs an investigation process which generates primary data each time as is performed to the decision-makers. One example is the in vivo testing of a new medicine. A pharmaceutical company applying for the approval of a medicine may need to convince different groups of physicians, chemists, and pharmacologists within the FDA that the medicine is effective. The company chooses tests to perform and truthfully reports the generated data. Test results are realized signals from a probability distribution depending on the effectiveness of the medicine. The FDA's science board members observe both the test procedure and data before voting for a final decision. Other examples include financial advising for a managerial board or consumer testing before launching a product[1].

These situations usually involve a biased agent, who wants the decision–makers to make the same decision regardless of the actual state of the world. Despite the truthful–revelation restriction on the agent's report, she may design the investigation process strategically in light of her preference. Kamenica and Gentzkow [2011a] examine the case of persuasion between one self–interested agent ("sender" hereafter) and one decision–maker ("receiver" hereafter). They show that the sender can be strictly better off than without persuasion by committing to a persuasion mechanism which generates noisy signals to the receivers. They also point out that the result does not easily extend to a multiple–receiver situation in which the receivers care about each other's decisions[2].

This paper investigates the role of persuasion mechanisms in a committee's collective decisions. We extend the Bayesian persuasion mechanism to a multi–receiver framework. Regarding Kamenica and Gentzkow's open question, our results show that the information transmission channel plays a crucial role in the attainability of the sender's optimal payoff. We compare public persuasion, in which all receivers observe the sender's choice of the mechanism and the generated signals simultaneously, to private persuasion, in which only the sender's mechanism is commonly known while each receiver gets a separate observation. We show that, under public persuasion, the sender can always achieve the concave closure of the set of expected payoffs, regardless of the number of signal realizations or voting rule.

[1] A survey by DellaVigna and Gentzkow [2009] summarizes the empirical studies on persuasion. Examples include but are not limited to persuading consumers of merchandise, persuading voters before elections, persuading donors to NPOs or charities, or persuading investors on financial markets. This survey also discusses a persuader's incentives and roles, such as advertisers, financial analysts, and the media. Some persuasion channels are public while others are private; for instance, newspaper advertisements target all ages while propaganda via the internet mainly attracts young citizens.

[2] See Kamenica and Gentzkow [2011a] Section 7.2.

The sender is weakly worse off under private persuasion. From the sender's perspective, a combination of both types of persuasion is also worse than a pure public one. In fact, any information channel into which some private elements are introduced yields the sender less than her optimal payoff. In the in vivo testing example, the pharmaceutical company may have the choice to run a series of tests for all scientist groups of the FDA board publicly, or one separate test for each group involved privately. Our finding suggests that the company can guarantee a strictly positive probability of an approval with the former arrangement. Yet it is not true with the latter arrangement[3].

Previous studies on public versus private communication largely focus on cheap talk models (Farrell and Gibbons [1989], Goltsman and Pavlov [2011]). We compare the information transmission channels by incorporating the sender's signal generating process. Specifically, we examine the sender's use of Bayesian persuasion mechanism, which has a direct impact on the formation of receivers' posterior beliefs. The key difference between the two types of persuasion lies in the posterior beliefs. Signals from public persuasion lead to a common belief about the unknown state of the world for all receivers. The formation of the common belief is directly controlled by the sender's choice of a persuasion mechanism. The mechanism generates noisy signals to persuade just enough receivers to vote informatively upon observing the "favorable" signal realizations. In contrast, under private persuasion, receivers' posterior beliefs are subject to a second–order uncertainty. A rational receiver forms his posterior belief by incorporating not only his private observation, but also the inferred distribution of signal realizations from other receivers' informative votes, conditional upon his vote being pivotal. As a result, it is more difficult for the sender to use the persuasion mechanism to influence receivers' posterior beliefs. There exists multiple equilibria; the sender's equilibrium payoffs from all the equilibria are weakly dominated by that from public persuasion.

Meanwhile, we also demonstrate that the sender's payoff ranking is robust to the number of signal draws. It provides insight into a question regarding the design of testification institutions: if the sender is to run a series of tests and to report the generated data, would she prefer presenting the whole set of data to all receivers publicly, or showing data from each separate test to each receiver privately? The answer is the former. Furthermore, if a biased designer is to set up both the signal-generating mechanism and the channel through which the generated information is presented, she would establish a public persuasion mechanism. This is true for any number of tests or any amount of data that might be

[3] One may argue that it increases the sender's cost to prepare different information packages for different people. Nevertheless, we show that, even *without* the additional cost in approaching the receivers separately, the sender still weakly prefers public persuasion.

required by the testification procedure. The reason is that the sender adjusts the mechanism accordingly if she anticipates a more thorough examination with a series of trials. Because of receivers' common belief under public persuasion, multiple trials are more likely to help the receivers to discover the true state, including the sender's least–favored state. To offset this negative effect, the sender's optimal mechanism becomes less informative and generates noisy signal realization with greater probability in the state where players' interests are mis–aligned.

Finally, we derive a condition under which the receivers are weakly better off under private persuasion. Given an m–majority voting rule, when the cutoff receiver's preference does not exceed the average preference of all others, receivers' welfare from private persuasion weakly dominates that from public persuasion. It results from the strategic voting effect under private persuasion. Besides his own signal observation, a typical receiver incorporates additional information revealed by others' equilibrium votes. This second–order belief regarding each other's information helps receivers to reduce decision error and to improve welfare. On the other hand, however, the receivers' benefit from private persuasion is also limited. The welfare-enhancing effect of private information channel diminishes if the cutoff receiver's preference is beyond a certain threshold level. Under such a circumstance, a receiver interprets the differences in each other's votes more as a consequence of heterogeneous preferences but less as a result of different private signal observations. Particularly, our result indicates that private persuasion may become socially undesirable if the condition of receivers' preferences is not satisfied. In such a case, both the receivers and the sender are (weakly) better off under public persuasion.

The remainder of the paper is organized as follows. Section 1.1 reviews the related literature on persuasion games and voting. Section 2 introduces the model, strategies, and equilibrium concept. Section 3 characterizes the subgame perfect equilibria of the persuasion game under public and private persuasion, respectively. There exists a unique equilibrium under public persuasion, while under private persuasion there are multiple equilibria, some of which involve a sufficiently large portion of receivers voting uninformatively. Section 4 shows that the sender is weakly better off under public persuasion. In addition, Section 4.2 demonstrates that sender can always achieve the upper bound of the set of her expected payoffs via public persuasion. Section 5 provides a condition under which receivers are weakly better off under private persuasion. Increasing the number of signal draws under public persuasion does help the receivers to reduce decision error. Section 6 extends the model to a continuous signal realization space, and Section 7 concludes.

1.1 Related Literature

Farrell and Gibbons [1989] are among the first to provide discussions about public versus private information transmission. They analyze a one-sender two-receiver cheap-talk game with binary state space. They focus on the informativeness of information transmission and show that, whenever there exists an equilibrium in which the sender communicates informatively under private communication, there is an equilibrium under public communication in which the sender does the same[4]. Farrell and Gibbons [1989] do not emphasize the welfare implication of the communication environment. We show, in addition to Farrell and Gibbons [1989], that the sender always achieves a higher expected payoff under public persuasion.

There are three key differences between Farrell and Gibbons' setting and ours. First, the players' conflicts of interest are different. Farrell and Gibbons [1989] adopt a cheap-talk model in which not only the receivers', but also the sender's utility is state-dependent. We analyze a persuasion game in which only the receivers prefer state-contingent decisions; the sender always intends to induce one decision regardless of the state. Second, we analyze different communication techniques. In Farrell and Gibbons [1989], the sender uses costless and non-verifiable messages, whereas in our model, the sender has no direct control over what the receivers observe, though she can choose the signal-generating mechanism. Third, the receivers follow different decision rules. In Farrell and Gibbons [1989], each receiver takes an action separately, with no interaction of any form occurring between them. In our setting, the final decision is made through q-rule voting. The sender has to incorporate the strategic interactions between receivers' beliefs. The rest of this section will discuss related literature on each of the three features: sender's state-independent preference, persuasion mechanism, and strategic voting.

The conflict-of-interest feature of our model, which incorporates sender's state-independent preference, stems from the traditional persuasion game literature. Grossman [1981] and Milgrom [1981] analyze the information disclosure via verifiable messages between a seller (sender) and buyers (receiver). Although babbling equilibria in which the receiver ignores any of the sender's messages always exists, in a sequential equilibrium, a high-quality seller distinguishes herself by making a full disclosure. This is because such equilibria imposes restrictions on the receiver's off-equilibrium beliefs; the receiver forms rational expectation about the sender's true types. Chakraborty and Harbaugh [2010] investigate a sender's

[4]Goltsman and Pavlov [2011] extend Farrell and Gibbons [1989] to continuous state space and continuous action space. They also show that the sender is willing to reveal more information if she is allowed to combine the two messaging channels together.

persuasion through cheap-talk messages. With one-dimensional state space and state-independent preferences, no non-babbling equilibrium exists in the cheap-talk model. In contrast, with a multidimensional state space, a sender can convince a receiver by making credible comparative statements over the two states. Nonetheless, when there are two receivers to make the decision, interactions between them might offset the benefit from the sender's persuasion.

The sender's information generating process of our model is built on recent development in the literature of persuasion mechanisms. We extend Kamenica and Gentzkow [2011a] to a multi-receiver framework. Kamenica and Gentzkow [2011a] analyze a Bayesian persuasion game between one informed sender and one uninformed receiver. The sender chooses a state-dependent persuasion mechanism to provide signal realizations to the receiver. Both the mechanism and the generated signal are known to the receiver. They derive the necessary and sufficient conditions under which an optimal mechanism that yields the sender strictly positive benefit exists[5]. The key difference between Kamenica and Gentzkow [2011a] and our study arises from the strategic feature of the receivers' collective decision-making. In Kamenica and Gentzkow [2011a], the sender faces a single receiver, who updates his belief upon observing one signal realization and makes a decision on his own. The key issue of Kamenica and Gentzkow [2011a] is whether the sender can commit to a persuasion mechanism so that the receiver will take the sender's preferred action upon receiving a more favorable signal. In our model, multiple receivers vote for one final decision. A receiver makes an implicit inference about the distribution of others' signal observations conditional on his own vote being pivotal. In turn, the sender incorporates this strategic effect into her choice of mechanisms. The sender continues to face the problem of convincing enough receivers to vote for the sender-preferred alternative upon observing a favorable signal, but each receiver's incentive to do so changes[6].

[5]Two more papers discuss similar mechanisms. Rosar and Schulte [2010] look into the design of a device with which an imperfectly informed sender can generate public information about the underlying state. The set of the generated information is a superset of the set of the sender's private information. However, the designer's goal is to provide information as precise as possible. Unlike in Kamenica and Gentzkow [2011a] and our study, the sender is biased towards one decision outcome. Rayo and Segal [2010] develop a sender's optimal disclosure rule with a multidimensional state space. In Rayo and Segal [2010], the receiver also has private information regarding the true state; yet in our model, the receivers do not possess any relevant information.

[6]The discussion of "persuasion mechanism" is also related to a broader scope of literature on persuasion rules. Glazer and Rubinstein [2004] study a persuasion game with one sender and one receiver, the latter of whom can verify the former's report for at most one of two aspects. The persuasion mechanism, with the objective to minimize the probability of the receiver's decision errors, specifies a set of cheap-talk messages for the sender to choose from, a device for the receiver to select the aspect to be checked, and a rule for the receiver to take the final action. Glazer and Rubinstein [2006] examine a similar setting but allow the receiver to randomize at the final stage of the decision. Glazer and Rubinstein [2006] show that there exists a persuasion rule with no randomization, and all optimal rules satisfy ex-post optimality.

Third, on comparing the two communication environments, we emphasize the strategic voting effect of receivers upon observing private signal realizations. The existing voting literature has extensively discussed how private information affects the quality of collective decisions. Austen-Smith and Banks [1996] (and Feddersen and Pesendorfer [1998]) point out that, as a challenge to the Condorcet Jury Theorem, it is not rational for each voter to vote only according to his private signal. A rational voter has to take into account the information revealed by the event of one's vote being decisive[7]. Feddersen and Pesendorfer [1997] analyze a general voting model in which voters have heterogeneous preferences and receive noisy private signals from different information services. Feddersen and Pesendorfer [1997] fully characterize the voters' equilibrium voting behavior and demonstrates that a q-rule voting fully aggregates information despite the fact that the fraction of voters who vote informatively decreases to zero as the electorate grows to infinity. Gerardi and Yariv [2007] considers a committee voting with deliberation. Sequential equilibria exists with committee members truthfully revealing their own private information and rationally adjusting their votes according to other members' information. In this paper, the information service from which private noisy signals are generated is from the sender. The novelty is that the sender optimally chooses from a family of information service and optimally adjusts the strength of the signals in each state.

The mechanism design problem of our model is close to that of Caillaud and Tirole [2007], who examines a one-sender multiple-receiver model. But the "persuasion mechanism" examined in our setting is different from Caillaud and Tirole [2007]. Their paper focuses on how an uninformed sender could sequentially persuade a group of receivers for the approval of a project. The mechanism selects key receiver(s) to conduct a costly investigation. The sender does not possess any relevant information regarding the underlying state. Our paper looks into the "investigation process" itself, which has not been elaborated by Caillaud and Tirole [2007]. We show that, as long as the sender can design the investigation process, the difference in information transmission environments matters even though the receivers have no investigation cost and are willing to examine the sender's report.

Moreover, the multi-receiver setting of our model is close to Taneva [2014], who also investigates a designer's optimal choice of information structure subject to agents playing a Bayesian Nash equilibrium. She provides a complete characterization of the informational

[7] In defense of the Condorcet Jury Theorem, McLennan [1998] shows that, for any common interest voting game, if with the sincere voting assumption collective decisions can successfully select the right alternative, there exists an equilibrium voting profile with everyone responding strategically. More importantly, outcomes from all such Nash equilibria are at least as good as the ones from sincere but non-optimal voting profiles.

design problem with agents interacting a symmetric binary-state 2×2 normal-form game. Our paper examines a sender's optimal information revelation problem prior to a voting game with any number of players. Our model incorporates each voter's second-order belief regarding others' signal observations under private signals, while Taneva [2014] does not consider such higher-order belief effect. Our results show that the second-order uncertainty in players' beliefs serves as a key factor for the outcome difference between public and private persuasion.

2 Model

2.1 Setup

We analyze a Bayesian persuasion model with one sender ("she") and n receivers ("he"). Players' payoffs depend on the state of the world $t \in T = \{\alpha, \beta\}$ and the receivers' collective decision. We assume that the common prior probability distribution over T are $\text{prob}(\alpha) = p, \text{prob}(\beta) = 1 - p$, where $p \in [0, 1]$. The collective decision is determined by voting, with alternatives denoted by $\{A, B\}$. Each receiver i casts a vote $v_i \in \{A, B\}$. Votes are aggregated by q-rule, which characterizes the minimum number of votes $m \in \{1, ..., n\}$ needed to implement alternative B[8]:

$$v(v_1, ..., v_n; m) = \begin{cases} B, & \text{if } |\{j : v_i = B\}| \geq m; \\ A, & \text{otherwise.} \end{cases}$$

Let $u^S(v, t)$ and $u_i^R(v, t), i = 1, ..., n$ denote the utility that the sender and each receiver derive from the implementation of the collective decision v in state t, respectively. We assume the sender's utility is state-independent:

$$u^S(B, t) > u^S(A, t), \forall t \in T$$

which means the sender always prefers alternative B being implemented regardless of the state t. The receivers, on the other hand, have state-dependent utilities.

$$u_i^R(B, \beta) > u_i^R(A, \beta), u_i^R(A, \alpha) > u_i^R(B, \alpha), \forall i = 1, ..., n$$

[8]We discuss all possible q-rules, where $q = \frac{m}{n}$, including simple majority ($m = \frac{n+1}{2}$), super-majority ($\frac{n+1}{2} < m < n$), and unanimity (m=n). Notice that, under the q-rule, the final decision is A if and only if at least $n - m + 1$ receivers vote for A. A receiver considers others' votes to be a tie when there are $m - 1$ votes for B and $n - m$ votes for A.

Receiver i prefers alternative A in state α and B in state β. And we allow heterogeneous preferences for different receivers, who might derive different levels of utility from each of the implemented alternatives and assign different levels of utility loss to incorrect decisions. Namely, given the state t and final decision v, $u_j^R(v,t) \neq u_i^R(v,t)$, for $j \neq i$. Notice that all players' payoff structures are common knowledge; nevertheless, the receivers do not observe the true state of the world.

The timing of the game is as follows. The sender sets up a signal-generating mechanism, which consists of a family of conditional distributions $\{\pi(\cdot|t)\}_{t\in T}$ over a space of signal realizations $S = \{a, b\}$[9]. The receivers get informed of $\{\pi(\cdot|t)\}_{t\in T}$. Then Nature determines the true state t, which is privately observed by the sender. The sender applies the conditional distribution in state t to generate noisy signal(s) and truthfully reveals the generated signal(s) to the receivers. Upon observing the signal(s), receivers cast votes simultaneously, and the final decision is determined by a q-rule specified above. The timeline of the game is illustrated in Figure 1.

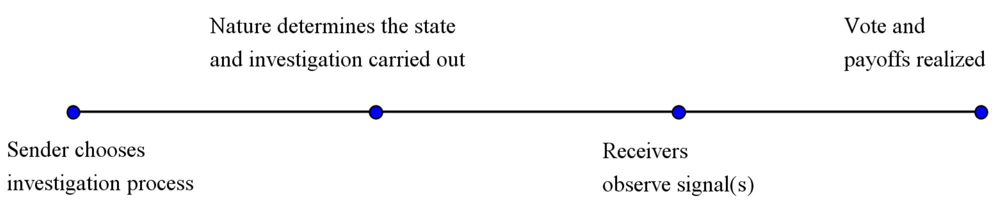

Figure 1: The Timeline of the Game

Note that different from traditional persuasion-game models, the sender does not have direct control over what the receivers might observe; instead, she tries to influence the receivers' decision by setting up a signal-generating mechanism. This can be interpreted as the revelation of the whole investigation process, such as conducting experiments, generating primary data, or running tests. Both the investigation process being employed and the evidence emerged are communicated truthfully to the decision-makers.

We compare the impacts of the sender's persuasion in two different institutions. Specifically, we are interested in the channel through which the generated signal realization(s) are transmitted once the persuasion mechanism is established. We investigate two types of persuasion:

- Public persuasion: the sender's choice of $\{\pi(\cdot|t)\}_{t\in T}$ is commonly known to all receivers. The generated signal realization $s \in S$, is also public information. All

[9]Notice that the sender chooses this set of conditional distributions before she observes the true state. We generalize the signal realization space to $S = [0, 1]$ in Appendix 6.

receivers observe the same signal.

- Private persuasion: the sender's choice of $\{\pi(\cdot|t)\}_{t\in T}$ is commonly known to all receivers. The signal realizations, drawn from the mechanism independently, are observed by each receiver separately and privately. A receiver will observe either $s = a$ or $s = b$, but not both.

2.2 Strategies and Equilibrium

This section specifies each player's strategy and the receivers' beliefs upon being informed of the signal-generating mechanism and observing the signal realization.

- The sender's strategy is to choose a family of conditional distributions $\{\pi(\cdot|t)\}_{t\in T}$ from $\mathcal{F} = \{\{\pi(\cdot|t)\}_{t\in T}|\pi : T \to \Delta S\}$, each element of which is a mapping from the state space to the simplex over the signal realization space.
- Each receiver's strategy, $\sigma_i : \mathcal{F} \times S \to \Delta\{A, B\}$, is a mapping from the Cartesian product of the collection of conditional distributions and the signal realization space to a simplex over voting alternatives[10].
- Each receiver's belief, $\mu_i : \mathcal{F} \times S \to \Delta T, i = 1, ..., n$, is a mapping from the Cartesian product of the collection of conditional distributions and the signal realization space to a simplex over the state space.

Similar to in Kamenica and Gentzkow [2011a], we adopt *subgame perfect equilibrium* as the equilibrium notion here:

Definition 1. *We define the equilibrium* $((\pi^*(\cdot|\alpha), \pi^*(\cdot|\beta)), \sigma_1^*, ..., \sigma_n^*)$ *of the multi-receiver Bayesian persuasion game as a subgame perfect equilibrium with no weakly dominated strategies:*

- *Given* $(\sigma_1^*, ..., \sigma_n^*)$, $\forall t \in T$, $(\pi^*(\cdot|\alpha), \pi^*(\cdot|\beta))$ *is the maximizer of* $EU^S(v, t)$
- *Given the sender's choice of* $(\pi^*(\cdot|\alpha), \pi^*(\cdot|\beta))$,

[10]It is worth noting that the strategy space in our setting is different from the one(s) in cheap-talk games (Crawford and Sobel [1982], Green and Stokey [2007]). In the latter, the sender also chooses a family of signalling rules, which specifies a probability distribution over the message space in each state. The key difference is that, in cheap-talk models, the receiver(s) only observes the realized message(s), not the sender's choice of the family of signalling rules. But in our model the receivers observe both.

- *Each receiver's vote,* $\forall i \in \{1,...,n\}, \forall s \in S$, $\sigma_i^*(\pi^*(\cdot|t), s) = \arg\max_{v_i} E_{\mu_i^*}(U_i^R(v,t))$, *where* v *is aggregated from* $v_i, i = 1,...,n$ *via a q-rule*

- *Each receiver's posterior belief derives from the above-specified strategies via Bayes' Rule, i.e.* $\mu_i^*(t|s) = \frac{\pi(s|t)p(t)}{\sum_{t' \in T} \pi(s|t')p(t')}$, *where* $p(t)$ *denotes the common prior over* T

- *None of the receivers use weakly dominated strategies.*

We assume that a receiver votes for B when he is indifferent between the two alternatives under the measure of his posterior belief μ_i. Thus, with a binary signal realization space, we can restrict our attention to pure strategy voting profiles with a typical form $(v_1^*, ..., v_n^*)$. More discussion about the voting behavior when space S is continuous is in Section 6.

A useful remark here is that in a subgame perfect equilibrium defined above, the exclusion of receiver i's weakly dominated strategies implies receiver i's adoption of a "threshold voting" strategy. Denote i's posterior beliefs $\mu_i^\alpha, \mu_i^\beta$ for any given signal realization s generated by $\{\pi(\cdot|t)\}_{t\in T}$. This receiver's expected payoffs from voting for each of the alternatives are:

$$B : \mu_i^\beta \cdot u_i(B,\beta) + \mu_i^\alpha \cdot u_i(B,\alpha)$$
$$A : \mu_i^\beta \cdot u_i(A,\beta) + \mu_i^\alpha \cdot u_i(A,\alpha)$$

where $\mu_i^\beta + \mu_i^\alpha = 1$ for any given s. We say receiver i adopts a *threshold voting strategy* if he votes for B if and only if

$$\mu_i^\beta \geq \frac{u_i(A,\alpha) - u_i(B,\alpha)}{u_i(A,\alpha) + u_i(B,\beta) - u_i(B,\alpha) - u_i(A,\beta)} \doteq q_i$$

We define, for receiver i, a *threshold doubt* q_i as a threshold value for the posterior belief above which this receiver will vote for B. We write the order statistics of the receivers' threshold doubts as $q_1 \leq ... \leq q_n$ and relabel the corresponding receivers as $R_1, ..., R_n$.

Non–weakly–dominated–strategy implies threshold voting for the following reason. Suppose receiver i does not adopt the threshold voting strategy. Suppose he observes realization s and his posterior belief μ_i^β is greater than the threshold doubt q_i; yet he votes for A. If this is part of an equilibrium in which i's vote is not pivotal, then given other receivers $-i$'s strategies in such an equilibrium, receiver i cannot be worse off by voting for B instead. And among all other strategy profiles of other receivers $-i$, this receiver i's vote will be pivotal given some strategy profiles of $-i$'s. In this case, receiver i is strictly better off by voting for B instead. Thus, when receiver i has posterior belief exceeding his threshold doubt (i.e. $\mu_i^\beta > q_i$), voting for B weakly dominates voting for A. Therefore, in any subgame perfect equilibrium that excludes the use of weakly dominated strategies, a receiver adopts the threshold voting strategy.

The above argument also implies, when one's vote is *not pivotal*, a receiver still adopts the threshold-voting strategy. This means receiver i votes for one of the alternatives by comparing his q_i with posterior belief μ_i^β given his own signal realization s. Fix a voting rule m/n, a receiver's vote is *not pivotal* when at least an $m-$majority vote for B, or at least an $n - m + 1$-majority vote for A. Suppose in the voting subgame, all other receivers vote for A; thus receiver i is not pivotal. For any signal realization s, suppose the corresponding posterior belief $\mu_i^\beta \geq q_i$. This receiver i could have voted for A regardless of his belief, given all others' voting behavior and the fact that his own vote could never be decisive. Nevertheless, as argued above, voting for A regardless of one's posterior belief μ_i^β is a weakly dominated strategy, which is excluded from the equilibrium concept defined above.

Yet it is possible that the all–receivers–vote–for–A strategy profile constitutes an equilibrium in the voting subgame. Every receiver votes for A not because one's own vote is never pivotal in such a situation; rather, it is sequentially rational when everyone follows the threshold-voting strategy. Suppose receiver i's posterior belief satisfies $\mu_i^\beta < q_i$ given the signal realization s when all other receivers vote for A. Then the use of non–weakly–dominated–strategies implies this receiver voting for A, which is in accordance with the threshold-voting strategy.

2.3 Preliminaries

This section provides preliminary analysis for the receivers' and the sender's problems, respectively. Two assumptions are specified before we proceed to the equilibrium characterization. One is on the receivers' preferences; the other is the *monotone likelihood ratio property* of the sender's signal-generating mechanism.

First, as assumed above, the sender prefers B to be implemented regardless of the state t. To demonstrate the sender's net benefit from either type of persuasion, we focus on the more interesting scenario in which the receivers' collective decision without the sender's persuasion is always $v = A$. This means that at least $n - m + 1$ receivers' threshold doubts are sufficiently high such that they would always vote for A based on their common prior.

Formally, we assume:

Assumption 1. *Without the sender's persuasion, if the receivers were to cast a vote based on the common prior probability distribution over T, there will be less than m votes for option B, i.e.*

$$(1-p)\cdot u_i(A,\beta)+p\cdot u_i(A,\alpha)>(1-p)\cdot u_i(B,\beta)+p\cdot u_i(B,\alpha), i\in\{m,m+1,...,n\}$$

This is equivalent to a more convenient notation, $1-p<q_i$, for all $m\leq i\leq n$. We call A the receivers' *default* collective choice.

Next, we simplify the sender's problem. The sender derive utilities from each of the final decision $v\in\{A,B\}$ as:

$$EU^S=u^S(B,t)\cdot\text{Prob}(v=B)+u^S(A,t)\cdot\text{Prob}(v=A),\forall t\in T$$

where $\text{Prob}(v=A)=1-\text{Prob}(v=B)$ for each t. Thus the sender's problem reduces to choosing $\{\pi(\cdot|t)\}_{t\in T}$ to maximize the probability that the voting outcome is B, since:

$$\max_{\{\pi(\cdot|t)\}_{t\in T}} EU^S=u^S(A,t)+(u^S(B,t)-u^S(A,t))\cdot\text{Prob}(v=B),\forall t\in T \tag{1}$$

where all other terms besides $\text{Prob}(v=B)$ are constant. Note that the voting rule and Assumption 1 imply that this event occurs only when at least m receivers are persuaded to cast their votes to the non-default alternative B after observing the mechanism's signal realization(s).

The following assumption restricts our attention to a specific family of signal-generating mechanisms:

Assumption 2. *The signal-generating mechanism $\{\pi(\cdot|t)\}_{t\in T}$ satisfies* ***monotone likelihood ratio property*** *(MLRP thereafter) if*

$$\frac{\pi(s|t)}{\pi(s|t')}\geq\frac{\pi(s'|t)}{\pi(s'|t')}$$

for every $s>s'$ and $t>t'$[11]*.*

With a binary state space $T=\{\alpha,\beta\}$, the sender's choice of a family of conditional

[11]Notice that a mechanism $\{\pi(\cdot|t)\}_{t\in T}$ with $\frac{\pi(s|t)}{\pi(s|t')}=\frac{\pi(s'|t)}{\pi(s'|t')}$ generates each signal $s\in S$ with the same probability in every state. It can be easily shown that the receivers vote according to the prior probability distribution in this case.

distributions $\{\pi(\cdot|t)\}_{t\in T}$ is equivalent to finding densities $(\pi(s|\alpha), \pi(s|\beta))$. Assumption 2 implies the CDF $\Pi(\cdot|\beta)$ *first order stochastic dominates* $\Pi(s|\alpha)$, i.e. $\Pi(s|\beta) < \Pi(s|\alpha), \forall s \in S$[12]. The relative likelihood of generating a favorable signal from state β is larger than that from the other state.

Before proceeding to the main results of the paper, we have two remarks on the model setup. First, it does not matter whether the sender first chooses the investigation or Nature determines the true state, as long as the investigation process is truthfully revealed to the receivers and is verifiable ex-post. The sender's investigation specifies a family of conditional distributions over all possible states. Second, it is crucial that the investigation is reported to the receivers and is verifiable *ex-post*. The model reduces to a cheap-talk framework if the receivers only observe the generated signal realizations but can never discover the signal-generating mechanism. In that case, the sender would adopt a mechanism that would generate signal $s = b$ with probability 1 in both states, and the receivers would rationally ignore any signal observation in equilibrium.

3 Equilibrium Characterization: Public versus Private Persuasion

In this section, we first discuss receivers' voting behavior upon observing signal realizations, taking the sender's persuasion mechanism $\{\pi(\cdot|t)\}_{t\in T}$ as given. Then we solve the sender's optimization problem and characterize the equilibria of the persuasion game under public and private persuasion, respectively.

3.1 Receivers' Voting Behavior

Under either type of persuasion, once a persuasion mechanism $\{\pi(\cdot|t)\}_{t\in T}$ is chosen and signals realized, a typical receiver i updates his belief μ_i^β via Bayes Rule and casts his vote accordingly. It is helpful to define two concepts regarding receivers' voting strategies:

Definition 2. *A receiver* ***votes sincerely*** *when he maximizes expected payoff conditional on his own signal observation only.*

[12] A brief proof is included in the Appendix. And we will discuss the equilibrium characterization under this property more extensively in Section 6.

Definition 3. *A receiver votes **informatively** if his vote changes according to his **own** signal observation.*

Under Assumption 2, *informative voting* is equivalent to receiver i voting for B upon observing $s = b$ and voting for A upon observing $s = a$. And we call a voting strategy *uninformative* if a receiver votes for B (or A) regardless of his own signal realization.

Lemma 1. *Under public persuasion, if Assumption 1 and 2 hold, given any persuasion mechanism $\{\pi(\cdot|t)\}_{t\in T}$ the sender has chosen:*

- *All receivers vote sincerely according to the same signal observation;*
- *There exists cutoff values $\underline{q}_{\{\pi(\cdot|t)\}}, \bar{q}_{\{\pi(\cdot|t)\}} \in [0,1]$ with $\underline{q}_{\{\pi(\cdot|t)\}} < \bar{q}_{\{\pi(\cdot|t)\}}$ such that the receivers with threshold doubt $q_j \in [0, \underline{q}_{\{\pi(\cdot|t)\}}]$ vote for B uninformatively, the receivers with $q_k \in [\bar{q}_{\{\pi(\cdot|t)\}}, 1]$ vote for A uninformatively, and the rest vote informatively.*
- *The number of receivers who vote uninformatively for B is strictly less than m.*

The proof of this lemma is included in the Appendix. The first statement is obvious. Under public persuasion all receivers have the same signal observation. Each receiver casts vote conditional on belief updating upon this observation. The second statement indicates that there is a portion of receivers whose threshold doubts are too large to be convinced by a signal realization $s = b$. These receivers vote for A regardless of the signal observation. Note that if the number of these receivers exceeds $n - m$, the final decision remains as the default choice A. On the other side, there are receivers with sufficiently small threshold doubts; these receivers vote for B regardless of the signal observation. In addition, the third statement shows that there will not be too many such votes to shift the final decision to the non-default option B with probability 1.

Next, we describe the receivers' voting behavior under private persuasion, when each of them observes an independent signal realization separately. Sincere voting is no longer optimal. A rational receiver updates belief incorporating not only his own signal observation, but also the distribution of other receivers' observations. Such distribution can be inferred from the distribution of other receivers' informative votes, conditional on this receiver's own vote being pivotal. We define *strategic voting* as follows:

Definition 4. *A receiver **votes strategically** when he maximizes expected payoff conditional on his own observation and the event that his vote is pivotal.*

Note that one's vote being pivotal is the only situation in which his vote will ever affect the voting outcome and his utility. Given a voting rule m/n, the event of one's vote being pivotal indicates $m-1$ votes for B and $n-m$ votes for A among all others' votes. If a voting profile involves at least one other receiver voting informatively, receiver i can infer the distribution of other receivers' observations from the distribution of all informative votes. Denote $\gamma(k,r)$ the receiver's posterior belief μ_i^β when k out of r signals are $s=b$:

$$\gamma(k,r) = \frac{(1-p)\cdot\binom{r}{k}\cdot(\pi(b|\beta))^k(1-\pi(b|\beta))^{r-k}}{p\cdot\binom{r}{k}\cdot(\pi(b|\alpha))^k(1-\pi(b|\alpha)^{r-k}) + (1-p)\cdot\binom{r}{k}\cdot(\pi(b|\beta))^k(1-\pi(b|\beta))^{r-k}}$$

It is also worth noting that strategic voting does not imply informative voting, or vice versa. A receiver who votes uninformatively ignores his private observation rationally: upon observing either signal, a comparison between his updated posterior and threshold doubt might still lead to voting for one alternative regardless of his own observation.

The following proposition characterizes the receivers' sequentially rational voting profiles in the voting subgame:

Lemma 2. *Suppose Assumptions 1-2 hold and no receiver uses weakly dominated strategy in the voting stage. Then under private persuasion, when each receiver votes strategically,*

- (No-simple-approval) *the number of receivers who vote for B uninformatively is strictly less than m; the rest $n-m+1$ or more receivers either vote informatively, or vote for A uninformatively, and*
- (Multiplicity) $\exists$ *persuasion mechanism $\{\pi(\cdot|t)\}_{t\in T}$ which induces at least two voting profiles $(\tilde{v}_1, ..., \tilde{v}_n)$ and $(\hat{v}_1, ..., \hat{v}_n)$ with at least one receiver i's strategy differs $\tilde{v}_i \neq \hat{v}_i$; and both voting profiles satisfy sequential rationality given each receiver's posterior belief.*

Corollary 1. (Common-interest receivers) *When the receivers' preferences are sufficiently close, there exists no sequentially rational voting profile which involves some uninformative votes for B and some uninformative votes for A simultaneously.*

The proofs are included in the Appendix. Compared to Lemma 1, under private persuasion, the receivers become more skeptical when casting votes. This is because each receiver updates posterior belief upon the inferred distribution of signal realizations from all informative votes, not merely upon his own observation. Upon observing $s=a$, the event that one's vote is pivotal indicates some others' observing opposite realizations $s=b$; thus

μ_i^β is greater than its counterpart under public persuasion when receivers vote sincerely. On the other hand, upon observing $s = b$, one's vote being pivotal implies the presence of realizations $s = a$ from other receivers; thus, this receiver i becomes more "skeptical" when casting a vote for B.

We demonstrate Lemma 2 by screening over all possible voting profiles that could be induced in the voting subgame. Given any persuasion mechanism $\{\pi(\cdot|t)\}_{t\in T}$, voting profiles in the voting subgame can be classified into three classes:

Class 1 There are at least m receivers voting for B uninformatively;

Class 2 There are at least $n - m + 1$ receivers voting for A uninformatively; and

Class 3 There are $m - \ell$ receivers voting for B uninformatively, $n - m - k$ receivers voting for A uninformatively, and the rest $\ell + k$ receivers voting informatively, for $1 \leq \ell \leq m$ and $0 \leq k \leq n - m$.

As shown in the Appendix, Assumption 1 and 2 and the non–weakly–dominated–strategy requirement imply that no voting profile in *Class 1* satisfies sequential rationality. The first part of Lemma 2 immediately follows. It indicates that the sender can never implement option B with probability 1, whichever persuasion mechanism is chosen. Thus in any sequentially rational voting profile, there are strictly less than m receivers who vote for B uninformatively. We only need to consider voting profiles in *Class 2* and *Class 3* in the subsequent analysis. The second part of Lemma 2 examines multiplicity in the voting stage. In the Appendix, we construct a mechanism that induces either a voting profile from *Class 2*, or another profile from *Class 3*. And as we shall show in the next section, multiplicity affects the sender's optimal payoffs under private persusion.

Corollary 1 extends previous results to almost-common-interest receivers. When the receivers' preferences are sufficiently close to a common threshold doubt q, the number of sequentially rational voting profiles reduces. The remaining profiles include: all receivers voting uninformatively for A; or $n - m - k$ receivers voting uninformatively for A, $k + m$ receivers voting informatively ($0 \leq k \leq n - m$); or $m - \ell$ receivers voting uninformatively for B, $n-m+\ell$ receivers voting informatively ($1 \leq \ell \leq m$). Hence, when the receivers share almost–common interest, any voting profile consistent with sequential rationality does not allow the uninformative-A voters and the uninformative-B voters to exist at the same time.

3.2 Informativeness of Sender's Optimal Mechanisms

This subsection characterizes the optimal persuasion mechanism under each type of communication. Specifically, we investigate the sender's choice of conditional probabilities from which the signal realizations are drawn. We use the term *informativeness* to represent the probability that a mechanism generates a signal realization representing the current state.

3.2.1 Public Persuasion

Under public persuasion, all receivers observe the same signal realization. Since nobody has any private information, the distribution of their votes reveals no additional information regarding the true state. The final decision is B when at least m receivers vote for B upon observing a signal realization $s = b$. Re-write $EU^S = u^S(A,t) + (u^S(B,t) - u^S(A,t)) \cdot \text{Prob}(v = B), \forall t \in T$. Maximizing the sender's expected payoffs is equivalent to choosing $(\pi(b|\alpha), \pi(b|\beta)) \in \mathcal{F}$ to maximize the probability of the voting outcome being $v = B$:

$$\max_{\pi(b|\alpha),\pi(b|\beta)} p \cdot \pi(b|\alpha) + (1-p) \cdot \pi(b|\beta) \tag{2}$$

subject to the receivers' voting behavior described in Lemma 1. Then we have the following proposition:

Proposition 1. *Under public persuasion, the sender's optimal persuasion mechanism generates signal $s = a$ with positive probability in state α and generates signal $s = b$ with probability 1 in state β, i.e.*

$$\pi^*_{PUB}(a|\alpha) = \frac{q_m - (1-p)}{q_m \cdot p}, \pi^*_{PUB}(b|\beta) = 1$$

which holds for all $m \leq n$. Moreover, there are at most m receivers who vote informatively, i.e. $\bar{q} = q_m$.

The proof for this result is included in the Appendix. We make three remarks here. First, this proposition is true for all voting rules m/n, including majority rule $m = (n+1)/2$, super-majority rule $(n+1)/2 < m < n$, and the unanimity rule $m = n$. Second, under public persuasion there are $n - m$ receivers with the highest threshold doubts voting uninformatively for A. This is because the sender only needs to convince m receivers with smaller threshold doubts to vote for B upon observing an $s = b$ observation. More precise signals are needed to convince additional receivers, which lowers the sender's expected payoff. Third, a similar result holds for common-interest receivers, where the parameter q_m is replaced by a common threshold doubt q.

3.2.2 Private Persuasion

Under private persuasion, each receiver observes an independent signal realization separately. Compared to the case of public persuasion, the change in receivers' voting behavior affects both the constraint set and the objective function of the sender's optimization problem. The constraint set shrinks due to the receivers' strategic voting. As described in Section 3.1, each receiver becomes more "skeptical" upon observing his own realization. He also takes into account the distribution of others' observations whenever they vote informatively. The sender's objective function changes because she can no longer ensure that all receivers observe the same signal realization. The number of favorable $s = b$ signal realizations follows a Binomial distribution. The optimal persuasion mechanism needs to generate enough $s = b$ realizations among all informative voters.

The proof of Lemma 2 considers three classes of receivers' strategy profiles in the voting subgame. Lemma 2 excludes all profiles of *Class 1*. The remaining two classes of voting profiles and the corresponding sender's problems are:

Class 2 When at least $n - m + 1$ receivers voting for A uninformatively, the voting outcome remains as $v = A$. The probability of implementing option B is 0.

Class 3 When $m - \ell$ receivers vote for B uninformatively, $n - m - k$ receivers vote for A uninformatively, and the rest $\ell + k$ receivers vote informatively, $(1 \leq \ell \leq m$ and $0 \leq k \leq n - m)$, the sender chooses $(\pi(b|\alpha), \pi(b|\beta)) \in \mathcal{F}$ to maximize the probability of $v = B$

$$\max_{\pi(b|\alpha),\pi(b|\beta)} p \cdot [\sum_{i=\ell}^{\ell+k} \binom{\ell+k}{i} (\pi(b|\alpha))^i (1 - \pi(b|\alpha))^{\ell+k-i}]$$
$$+(1-p) \cdot [\sum_{i=\ell}^{\ell+k} \binom{\ell+k}{i} (\pi(b|\beta))^i (1 - \pi(b|\beta))^{\ell+k-i}]$$

subject to the receivers' voting strategies of *Class 3*[13]. The objective function is continuous,

[13]Note that the induced receivers' voting profiles in *Class 3* need not necessarily be symmetric (Definition 7). Nonetheless, as we show in Lemma 5 in the Appendix, for each ℓ and k, a persuasion mechanism that

and the constraint set compact. Therefore, the sender's optimization problem is well–defined. The following proposition characterizes the informativeness of the sender's optimal mechanism under private persuasion:

Proposition 2. *Under private persuasion with any voting rules m/n, when all receivers vote strategically, the sender's optimal persuasion mechanism generates signal realizations with probability $\pi^*_{PRI}(a|\alpha), \pi^*_{PRI}(b|\beta)$ that satisfy:*

$$\pi^*_{PRI}(a|\alpha) = 1 - \sqrt[m]{\frac{(1-q_m)\cdot(1-p)}{q_m \cdot p}}, \pi^*_{PRI}(b|\beta) = 1$$

The proof of this proposition is included in the Appendix. Similar to Proposition 1, we make two remarks here: first, this proposition is valid for all voting rules m/n. Second, a similar result holds for common-interest receivers, where the parameter q_m is replaced by a common threshold doubt q.

Proposition 2 characterizes the informativeness of the sender's persuasion mechanism in any subgame perfect equilibrium under private persuasion. Note that, for all the induced voting profiles of *Class 3*, the sender's expected payoff decreases in $\pi(a|\alpha)$ and increases in $\pi(b|\beta)$. The proposition indicates that, in state β, where sender's and receivers' preferences are perfectly aligned, the sender sets the persuasion mechanism with the highest precision; the signal realization $s = b$ which indicates the current state is transmitted without any added noise. On the contrary, in state α with conflict of interest between sender and receivers, the information precision declines. The noisy signal $s = b$ is generated with positive probability in state α. Nonetheless, the sender cannot reduce the information precision unboundedly. As a result of strategic voting, the signal $s = a$ which indicates the current state α has to be precise enough to convince enough receivers to vote informatively upon each observation.

The sender faces three effects of opposing directions when choosing between persuasion mechanisms. The first one is the **information precision effect**, which exists under both public and private persuasion. Since less than m receivers would vote for alternative B based on the common prior, the sender adopts a mechanism which could convince more receivers to vote for B upon observing a favorable signal $s = b$. Convincing more receivers with higher threshold doubts requires signals with better precision in state α. This in turn increases the probability that the receivers discover the true state and decreases the

induces a symmetric informative-voting profile yields the sender strictly higher expected payoff than a mechanism which induces asymmetric voting profiles.

sender's benefit from the persuasion.

The **Binomial distribution effect** results from the uncertainty in realized signals under *private persuasion*. Unlike public persuasion, receivers observe signal realizations separately and no longer form a common posterior belief. If the sender adopted a mechanism the same as in public persuasion, it would be less likely to implement option B: exact m receivers are persuaded by such a mechanism and vote for B upon observing $s = b$; yet B cannot be implemented unless all the m *i.i.d.* observations are $s = b$, which is a rare event. To offset such uncertainty, the sender could adjust the information precision to convince more receivers to vote informatively. The probability of at least m receivers voting for option B follows a Binomial probability distribution. It compensates the sender for the loss from the uncertainty associated with the independent signal draws.

The **strategic-voting effect** is an effect stemming from the receivers' strategic voting behavior under *private persuasion*. In addition to the randomization of signal draws associated with private persuasion, this effect adds another layer of uncertainty to receivers' posterior beliefs. As characterized in Section 3.1, a rational receiver forms his posterior belief, incorporating not only his own observation, but also the inferred distribution of signals from other receivers' informative votes, conditional upon the current vote being pivotal. This second-order uncertainty in receivers' beliefs further restricts the attainability of the upper bound of the set of sender's optimal payoffs.

We devote the rest of this section to discussing the multiplicity of equilibrium voting profiles under each type of persuasion. The following proposition shows that multiple equilibrium outcomes exist under private persuasion, while public persuasion has a unique equilibrium outcome.

Corollary 2. *Under public persuasion there is a unique subgame perfect equilibrium in which the sender establishes a mechanism that induces exact m receivers to vote informatively. On the contrary, under private persuasion, there exists other equilibria in which at least $n - m + 1$ receivers vote* ***uninformatively*** *for option A.*

The proof for this corollary is embedded in the proof for Proposition 1 and 2. We make two remarks here. First, for both types of persuasion, the corollary excludes the circumstances under which all receivers ignore their observations and vote for B uninformatively. Second, under private persuasion, it is possible for all receivers to vote for A uninformatively. As characterized in Lemma 2, there exists equilibrium voting profiles in which at least $n - m + 1$ receivers vote for option A regardless of their signal observations. However, such voting profiles are not part of the equilibrium profiles under public persuasion.

Public persuasion gives rise to a common belief of all receivers. From the sender's perspective, the only uncertainty lies in the realized signals. In the unique subgame perfect equilibrium, the sender adjusts information precision of the mechanism such that m receivers with lower threshold doubts vote informatively, while the rest $n - m$ receivers with more extreme preferences vote for the default option regardless of signal observation. By setting up such a mechanism, the sender ensures herself full control over the receivers' posterior belief upon each observation.

In contrast, the uniqueness result does not hold under private persuasion. There are two additional uncertainties when a receiver forms posterior belief: the uncertainty associated with an independent signal draw and the uncertainty resulted from his second-guess about the distribution of others' observations. Moreover, the probability of implementing option B depends on which voting profile is induced at the voting stage. As shown by Lemma 2, there exist multiple sequentially rational voting profiles induced by a persuasion mechanism; those in the category of *Class 2* yield the sender a payoff of 0, regardless of the realized signal draws. Specifically, we show that the mechanism (x^*, y^*) which induces the sender-optimal voting profile of *Class 3* can also induce voting profiles of *Class 2*, in which more than $n - m+1$ receivers vote for the default option A uninformatively. Due to the multiplicity, private persuasion does not guarantee the sender a strictly positive expected payoff. Because of the two additional uncertainties, it is less effective for the sender's signal-generating mechanism to influence receivers' posterior beliefs.

4 Sender's Welfare Effects

In this section we first compare the sender's benefit from public persuasion with that from private persuasion. Proposition 2 and 3 provides a complete characterization of equilibria of private persuasion, and shows that in none of the equilibria sender's expected utility exceeds that of public persuasion. Moreover, we demonstrate that the welfare ranking is robust to the number of signal draws. Lastly, we show a stronger and more general result that the sender can always achieve the upper bound of the set of her expected payoffs via public persuasion.

4.1 Comparing Sender's Expected Payoffs: Public versus Private Persuasion

We re-write the sender's expected utility as $EU^S = u^S(A,t) + (u^S(B,t) - u^S(A,t)) \cdot \text{Prob}(v = B), \forall t \in T$. The maximum is attained when $\text{Prob}(v = B)$ is maximized in each state. Given the optimal persuasion mechanism $(\pi^*(\cdot|\alpha), \pi^*(\cdot|\beta))$ under each type of persuasion, we have the following comparison result:

Proposition 3. *In equilibrium, the signal-generating mechanism under public persuasion yields the sender strictly positive benefit, while the one under private persuasion generates non-negative benefit compared to the case without persuasion. Moreover, the sender's expected utility under public persuasion is weakly higher than that under private persuasion:*

$$EU^S_{PUB} > u^S(A,t), EU^S_{PRI} \geq u^S(A,t), \text{ and } EU^S_{PRI} \leq EU^S_{PUB}$$

The proof of this result is included in the Appendix. Compared to the scenario without persuasion, the sender is weakly better off under private persuasion and is strictly better off under public persuasion. The probability of implementing option B differs: $\text{Prob}_{\text{PUB}}(v = B) > 0$ and $\text{Prob}_{\text{PRI}}(v = B) \geq 0$. The first inequality is strict because of the unique equilibrium outcome under public persuasion. The second weak inequality results from equilibrium multiplicity under private persuasion. A mechanism (x^*, y^*) that induces the optimal voting profile of *Class 3* may also induce a voting profile of *Class 2*, where at least $n - m + 1$ receivers vote for the default option regardless of their signal observations. In this case, the sender can never implement option B, and her expected utility remains at the minimum level $u^S(A,t)$. Admittedly, the sender could instead choose a mechanism $(\tilde{x}, \tilde{y})$ to avoid inducing the voting profiles of *Class 2*. However, as shown in the proof, the information precision in state α must increase, and the sender's expected payoff declines to less than that generated under public persuasion: $\tilde{\text{Prob}}_{\text{PRI}}(v = B) < (1-p)/q_m$. In summary, the maximum of sender's expected utilities under private persuasion may achieve, but would never exceed that of public persuasion.

In short, the sender is better off with persuasion mechanisms, but the benefit is weakly less under private persuasion. Due to equilibrium multiplicity under private persuasion, there is no guarantee for the sender to implement option B with strictly positive probability. In contrast, under public persuasion, receivers form a common belief, which is directly controlled by the sender's choice of mechanism. This result describes the sender's "cost" of communicating in a private environment: the receivers' strategic voting gives rise to a second-order uncertainty over posterior beliefs, which imposes further difficulties for the sender to control the voting outcome via signal-generating mechanisms.

4.1.1 Multiple Draws of Signals under Public Persuasion

In previous sections we have focused on public persuasion with one signal draw and private persuasion with n draws. A natural question is whether our results, especially the fact that private persuasion makes the sender weakly worse off, are driven by the disparity in the numbers of signal realizations. The answer is no, as the following part will soon clarify. The welfare comparison is robust to the number of signal draws. In fact, there is a unique equilibrium in which sender's expected utility achieves the same positive level as long as all signal realizations are publicly observable.

Consider the institution which requires the sender to choose a mechanism $\{\pi(\cdot|t)\}_{t\in T}$ to draw $\mathbf{n}$ independent signal realizations; all receivers observe these realizations publicly. Despite the number of trials, all other elements of the game remain unchanged. Just as in public persuasion with one signal draw, receivers hold a common posterior belief. Denote $\gamma(\ell, n)$ the posterior belief μ^b when there are ℓ signals $s = b$ out of all n signals. Assumption 2 implies that $\gamma(\ell-1, n) < \gamma(\ell, n), \forall \ell \in \{1, ..., n\}$. A receiver i with threshold doubt q_i votes for B when there are more than ℓ favorable signals ($s = b$) out of all n signals if and only if $\gamma(\ell-1, n) < q_i < \gamma(\ell, n)$.

The sender chooses a mechanism such that at least m receivers will vote for B upon observing more than a certain number ($\hat{\ell}$) of favorable signals, i.e. $\gamma(\hat{\ell}, n) \leq q_m \leq \gamma(\hat{\ell}+1, n)$. The sender's objective is to maximize the probability that at least $\hat{\ell}$ favorable signals are generated in each state:

$$\max_{\pi(b|\alpha),\pi(b|\beta)} p \cdot \sum_{j=\hat{\ell}}^{n} (\pi(b|\alpha))^j (1-\pi(b|\alpha))^{n-j}$$
$$+(1-p) \cdot \sum_{j=\hat{\ell}}^{n} (\pi(b|\beta))^j (1-\pi(b|\beta))^{n-j}$$
$$\text{subject to } \gamma(\hat{\ell}, n) \leq q_m \leq \gamma(\hat{\ell}+1, n)$$

Solving the sender's problem yields the following proposition:

Proposition 4. *Under public persuasion with multiple independent signal draws, the sender's* *optimal mechanism generates signal realizations with probabilities:*

$$\pi^*_{MD}(a|\alpha) = 1 - \sqrt[n]{\frac{(1-q_m)\cdot(1-p)}{q_m \cdot p}}, \pi^*_{MD}(b|\beta) = 1$$

Moreover, the sender's expected utility is the same as that under public persuasion with a single signal draw.

$$EU^S_{MD} = EU^S_{PUB}$$

The proof of this proposition is included in the Appendix. The first part can also be expressed as $\pi^*_{\text{MD}}(b|\alpha) = \sqrt[n]{\pi^*_{\text{PUB}}(b|\alpha)}$, and $\pi^*_{\text{MD}}(b|\beta) = \sqrt[n]{\pi^*_{\text{PUB}}(b|\beta)}$. Since $\pi^*_{\text{MD}}(b|\alpha) > \pi^*_{\text{PUB}}(b|\alpha)$, the optimal mechanism generates noisy signals with higher probability in state α when multiple signals instead of a single one are drawn. Compared to public persuasion with one signal draw, the signals become less precise in the state in which the players' interests are mis-aligned. Moreover, as $n \to \infty$, $\pi^*_{\text{MD}}(b|\alpha) \to 1$, which means the informativeness of the mechanism drops drastically as the number of trials grows.

The second part of the proposition shows that the sender achieves the same level of expected utility as she does under public persuasion with a single trial. The reason is that, as long as the information channel remains public, receivers form a common posterior belief. Since drawing multiple signals increases the probability of observing the unfavorable signal realizations $s = a$, the sender offsets the potential loss by reducing the information precision. Therefore, requiring the sender to reveal the results of n trials does not make her worse off.

It is also worth comparing public persuasion with n signal draws to private persuasion, as both have the same number of *i.i.d.* signal realizations. The only difference is that, in the former case, all receivers observe the same n realizations, while in the latter situation, each receiver observes a separate realization privately. For any voting rule m/n, we have the following comparison:

Corollary 3. *Compared to private persuasion, under public persuasion with n independent signal draws, the sender's optimal mechanism generates less precise signals in the state where the sender's and receivers' preferences are mis-aligned:*

$$\pi^*_{MD}(a|\alpha) < \pi^*_{PRI}(a|\alpha)$$

And the sender's expected utility of generating n signal realizations under public persuasion *weakly dominates that under private persuasion:*

$$EU^S_{MD} \geq EU^S_{PRI}$$

The first part of the corollary shows that, in state α, the informativeness of the sender's optimal mechanism is higher under private persuasion than that under public persuasion with n independent signal draws. When the sender knows that n trials would be examined publicly, she will optimally choose a mechanism that generates noisy signal realization with higher probability in state α where players' interests are mis-aligned.

The second part of the corollary indicates that the sender still weakly prefers public persuasion to private persuasion when the institution requires her to generate the same number of independent signal realizations as in the private environment. The sender has to reveal all the n realizations publicly and truthfully; notwithstanding, keeping the information transmission channel public helps the sender to achieve higher expected payoffs. In contrast, revealing n signal realizations privately to each receiver one at a time cannot guarantee the sender a maximum level of benefit.

Our results also provide an answer to the following question: suppose n pieces of evidence would be examined by a group of receivers, and the sender is allowed to choose a communication channel to present the evidence. Should the sender show the n pieces of evidence to all the receivers publicly, or each piece to each receiver privately and separately? Clearly, the sender prefers the former. Given the requirement of the institution, the sender would adjust the signal-generating mechanism before the receivers observe the realizations. Public persuasion, regardless of the number of signal draws, guarantees the sender a payoff as the upper bound that any equilibrium outcome under private persuasion could achieve.

4.2 Public Persuasion and the Upper Bound of Sender's Expected Payoffs

In previous sections, we have shown that the sender's expected utility is higher under public persuasion. In this section, we present a stronger result: public persuasion helps the sender to achieve the "highest possible payoff" for any given common prior p. Specifically, we drop Assumption 1 and show that the sender can always attain the concave closure of the set of her possible payoffs under public persuasion regardless of the number of signal draws.

The following figures illustrate the sender's sets of possible payoffs as a function of the common prior p. The blue line in Figure 2 represents the sender's expected payoff without persuasion. When at least m receivers' threshold doubts are sufficiently low, i.e. $p < 1 - q_m$, the collective decision is B even without the persuasion. So the sender gets $\text{Prob}(v = B) = 1$. However, when $p > 1 - q_m$, the sender gets $\text{Prob}(v = B) = 0$. Thus the sender's expected payoff is discontinuous at $p = 1 - q_m$.

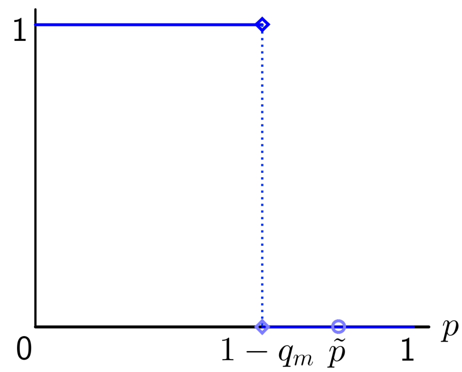

Figure 2: Sender's expected payoffs without persuasion

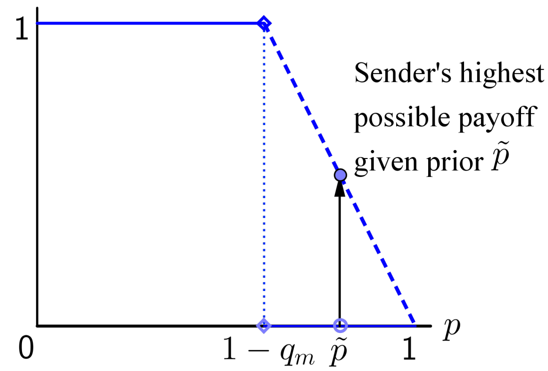

Figure 3: Sender's expected payoffs under public persuasion: the convex hull

The dashed blue line in Figure 3 illustrates the sender's payoff increment under public persuasion. To formally establish the result, we first define the *concave closure* of the graph of EU^S as:

Definition 5. *Denote* Λ *as the* concave closure *of* EU^S*:*

$$\Lambda(p) \doteq \sup\{u : (p, u) \in co(EU^S)\}$$

where $co(EU^S)$ *is the* convex hull *of the graph of* EU^S*.*

Then we have the following result:

Proposition 5. *For any voting rule* m *and any* $\tilde{p} \in [0, 1]$*, under public persuasion the sender achieves an expected payoff on the concave closure of the set of all possible payoffs, i.e.*

$$EU^S_{PUB}(\tilde{p}) = \Lambda(\tilde{p})$$

Moreover, this result remains unchanged when the sender is required to draw multiple signal realizations.

The proof of this result is included in the Appendix. Compared with the sender's payoff without persuasion in Figure 2, it is easy to see that, with public persuasion, $\forall \tilde{p} < 1 - q_m$, the sender's payoffs remain the same; while $\forall \tilde{p} > 1 - q_m$, the sender's expected payoffs lie on the dashed blue line in Figure 3. The probability of getting her preferred voting outcome is $\text{Prob}(v = B) = \frac{1}{q_m}(1 - \tilde{p})$. In other words, public persuasion helps the sender to achieve the concave closure of the convex hull of the sender's set of possible payoffs.

It is also worth noting that the uniqueness of the *supremum* guarantees the uniqueness of the sender's optimal payoff for each given $p \in [0, 1]$ under public persuasion. This is also true for the case in which multiple signal draws are generated. This is because signals under public persuasion always give rise to a common belief about the unknown state of the world. It does not matter whether all receivers observe a single signal draw or observe a series of signal draws. In this situation, the sender can appropriately choose a state-dependent persuasion mechanism to manipulate the common posterior beliefs. Her optimal mechanism generates noisy signals so that the posterior belief upon a signal realization $s = b$ is just above the m-th threshold doubt q_m.

Notice that the concave closure of the convex hull of the payoff set is the highest possible the sender is able to achieve. As shown in previous sections, adding any private element worsens the sender's welfare. A combination of public and private persuasion also makes the sender worse off.

4.2.1 Comparative Statics: Voting Rule and Sender's Expected Payoffs

At the end of this section, we present a comparative-static result of the sender's optimal payoff with respect to the voting rule m/n.

Proposition 6. *The sender's expected payoff from the optimal persuasion mechanism decreases as m/n, the proportion of votes required to implement option B, increases.*

The proposition results from the fact that $\text{Prob}(v = B)$ decreases as the "cutoff" receiver's threshold doubt increases from q_m to $q_{m+i}, \forall i > 0$. Intuitively, as the voting rule shifts from majority to super-majority, the sender needs to persuade more receivers with higher threshold doubts to vote for option B informatively. More precise signals are needed in order to convince these receivers. According to the *information precision effect*, the sender's optimal payoff from persuasion declines.

5 Receivers' Welfare

5.1 Receivers' Decision Errors and Welfare

In this section, we compare the receivers' welfare under public persuasion with that under private persuasion. Specifically, receivers' welfare is measured as the sum of utilitarian social welfare functions weighed by the probability of each type of decision error:

$$W = \frac{1}{n} \cdot \sum_{i=1}^{n} u_i(B, \alpha) \cdot \ \text{Prob}(v = B|\alpha) + \frac{1}{n} \cdot \sum_{i=1}^{n} u_i(A, \beta) \cdot \ \text{Prob}(v = A|\beta)$$

Without loss of generality, we can normalize receivers' utilities from correct decisions to $u_i(A, \alpha) = u_i(B, \beta) = 0$. Then the above expression becomes the weighted average utility losses from mistaken decisions. Given a voting rule m/n, term $\text{Prob}(v = B|\alpha)$ equals the probability of at least m receivers voting for B in state α, and term $\text{Prob}(v = A|\beta)$ equals the probability that at most $m-1$ receivers vote for B in state β so that the default option remains as the voting outcome.

Under public persuasion, the equilibrium uniqueness leads to a unique negative level of receivers' welfare. Under private persuasion, however, equilibrium multiplicity results in different levels of welfare, depending on the voting profile being induced in the specific subgame. As we shall demonstrate in the first part of the following proposition, receivers' welfare from all subgame perfect equilibria under private persuasion is no less than the welfare under public persuasion.

Moreover, if receivers suffer from a constant welfare loss under public persuasion with one signal draw, a natural question is whether their decision quality and welfare could improve when more independent signals are drawn and observed. Particularly, when the number of such independent trials increases to n, the number of signal observations under private persuasion, will the receivers' decision quality be as good as that under private persuasion? The second part of the following proposition provides a negative answer to this question.

Proposition 7. *Receivers achieve a weakly higher welfare level under private persuasion when the m-th receiver's threshold doubt q_m is below the arithmetic mean of all receivers' threshold doubts:*

$$W_{PRI} \geq W_{PUB} \text{ if and only if } q_m \leq \sum_{i=1}^{n} q_i/n$$

Moreover, the receivers' welfare under public persuasion with multiple signal draws is the same as that under public persuasion with a single signal draw:

$$W_{MD} = W_{PUB}$$

The proof of this proposition is included in the Appendix. The first part of the proposition shows that receivers are weakly better off under private persuasion as long as the value of q_m does not exceed the average of all others' q_i's. In terms of decision quality, the probability of decision errors is higher under public persuasion and lower under private persuasion. Note that the sender's and the receivers' interests are perfectly–aligned in state β and completely mis-aligned in state α, while the sender's optimal mechanism maximizes the probability of $s = b$ being generated in both states. In the unique equilibrium of public persuasion or the *Class-3* equilibrium of private persuasion, sender's optimal mechanisms maximize receivers' type I error $\text{Prob}(v = B|\alpha)$ while minimize type II error $\text{Prob}(v = A|\beta)$. Furthermore, receivers achieve a different welfare level if the uninformative voting profiles of *Class 2* is induced in equilibrium. In this situation, the sender's mechanism fails to manipulate the likelihood of either type of error. The probabilities of decision errors declines, and receivers' welfare improves as long as q_m is below the threshold level.

The second part of the proposition indicates that the receivers' welfare remains at the same level as the number of signal draws increases under public persuasion. Increasing the number of trials does not help receivers to reduce decision errors. The reason is that, as long as the persuasion channel remains public, receivers have no private information which might be revealed by others' votes in equilibrium. Drawing multiple signals increases the probability of the receivers observing unfavorable signals ($s = a$), and the sender will optimally adjust the information precision to avoid generating unfavorable draws too frequently.

In short, receivers make a better decision under private persuasion when the value of the m-th voter's q_m is relatively small. This result captures the receivers' "benefit" from communicating in a private environment: when voting strategically, receivers update posterior beliefs based not only on their own private signals, but also on additional information revealed by others' equilibrium votes. Under public persuasion, however, publicly observed signal draws lead to a common belief of all receivers, which excludes the possibility of inferring additional information from others' voting behavior.

On the other side, receivers' benefit from private persuasion is also limited: the welfare-improving effect of private environment does not exist if the m-th threshold doubt q_m exceeds the average threshold doubts of all receivers. In this situation, a receiver interprets the differences in each others' votes more as a consequence of heterogeneous preferences but less as a result of strategic voting with private signal observations. This result also indicates that private persuasion becomes socially undesirable if q_m is too high. Receivers can no longer benefit from smaller decision errors of private persuasion, while the sender always achieves higher welfare under public persuasion.

5.2 Voting and Information Aggregation under Private Persuasion

Under private persuasion, it is also worth examining the effectiveness of voting as a mechanism for aggregating receivers' private information. Traditional voting literature on information aggregation considers each voter privately observing a noisy signal from a neutral information service, usually Nature (Austen-Smith and Banks [1996], Feddersen and Pesendorfer [1998], McLennan [1998], Feddersen and Pesendorfer [1997]). In our setting, it is a biased sender who establishes an information service and generates noisy signals. As we have shown, the sender chooses from families of information service and optimally adjusts the precision of the signals in each state. We have the following result:

Corollary 4. *As the number of receivers $n \to \infty$, in the subgame perfect equilibrium which yields the sender the highest expected payoff, m-majority voting selects option B with probability 1 in state β, while the probability that m-majority rule fails to implement option A in state α is bounded away from 0.*

This corollary follows directly from Proposition 7. It indicates that voting fully aggregates the receivers' private information in state β, if not necessarily in state α. As characterized above, in the equilibrium where the sender's optimal symmetric voting profile is induced in the voting subgame, $\text{Prob}(v = B) = 1$ when $t = \beta$ and $\text{Prob}(v = A) < 1$ when $t = \alpha$. This is because the sender's optimal mechanism generates the most precise signals in the state where sender's and receivers' preferences are perfectly–aligned, whereas the mechanism generates noisy signal $s = b$ with strictly positive probability in the state where preferences are mis-aligned. As the size of committee grows, the sender adjusts the information precision such that the probability of implementing option B in state α remains strictly positive. In summary, full information aggregation may fail in some states if the information service is provided by a biased sender, whenever preferences are not perfectly–aligned.

6 Extension: Generalization of the Signal Realization Space

In this section we generalize the signal realization space to a continuous space, $S = [0,1]$. We characterize the receivers' voting behavior and then discuss the sender's optimal persuasion mechanisms in equilibrium.

A strategy for the sender is to choose a family of conditional distributions from $\mathcal{F} = \{\pi(\cdot|t)\}_{t\in T}$ each element of which maps from the state space T to the simplex over S. A strategy for receiver i is a measurable mapping $\sigma_i : \mathcal{F} \times S \to \Delta\{A, B\}$, where $\sigma_i(s_i;\pi)$ denotes the probability that this receiver votes for β upon observing signal s_i from $[0,1]$. Specifically, we are interested in a class of voting profiles where each receiver adopts a *cutoff strategy*:

Definition 6. *A receiver's voting strategy σ_i is called a* cutoff voting strategy *if $\exists \bar{s_i} \in [0,1]$ such that*

$$\sigma_i(s) = \begin{cases} 1, & \textit{if } s \geq \bar{s_i} \\ 0, & \textit{if } s < \bar{s_i} \end{cases}$$

In addition, we drop Assumption 1 and impose the following restriction on the sender's signal-generating mechanisms:

Assumption 3. *There exists $\hat{s} \in [0,1]$ such that $\forall s < \hat{s}$,*

$$\frac{\pi(s|\beta)}{\pi(s|\alpha)} < \frac{p}{1-p} \cdot \frac{q_m}{1-q_m}$$

Next we show how the cutoff signal $\bar{s_i}$ is determined and what properties it satisfies. Given a voting strategy σ_i, the probability that receiver i votes for B in state α is $\int_0^1 \sigma_i(s)\mu(a|s)ds$ and the probability that i votes for B in state β is $\int_0^1 \sigma_i(s)\mu(b|s)ds$. The probabilities that receiver i is pivotal in state α and in state β are as follows, respectively:

$$P_{-i}(\text{piv}|\alpha) = \sum_{|M|=m-1, i\notin M} (\prod_{j\in M} (\int_0^1 \sigma_i(s)\mu(\alpha|s)ds) \prod_{j\notin M, j\neq i} (\int_0^1 (1-\sigma_i(s))\mu(\alpha|s)ds))$$

$$P_{-i}(\text{piv}|\beta) = \sum_{|M|=m-1, i\notin M} (\prod_{j\in M} (\int_0^1 \sigma_i(s)\mu(\beta|s)ds) \prod_{j\notin M, j\neq i} (\int_0^1 (1-\sigma_i(s))\mu(\beta|s)ds))$$

For receiver i, define

$$\Delta(\sigma_{-i}, s) = P_{-i}(\text{piv}|\alpha) \cdot \pi(s|\alpha) \cdot q_i \cdot p - P_{-i}(\text{piv}|\beta) \cdot \pi(s|\beta) \cdot (1 - q_i) \cdot (1 - p)$$

Similar to Duggan and Martinelli Duggan and Martinelli [2001], we have the following lemmas:

Lemma 3. *Given the voting profile σ_{-i} of all other receivers, a strategy σ_i is a best response for receiver i if and only if*

$$\sigma_i(s) = \begin{cases} 1, & \textit{if } \Delta(\sigma_{-i}, s) < 0 \\ 0, & \textit{if } \Delta(\sigma_{-i}, s) > 0 \end{cases}$$

Moreover, it is equivalent to the following cutoff strategy

$$\hat{\sigma_i}(s) = \begin{cases} 1, & \textit{if } s \geq \bar{s_i} \\ 0, & \textit{if } s < \bar{s_i} \end{cases}$$

where $\bar{s_i} = \inf\{s \in [0,1] | \Delta(\sigma_{-i}, s) \leq 0\}$

Lemma 4. *The cutoff signal $\bar{s_i}$ is monotone increasing in the receiver's threshold doubt q_i. Moreover, the cutoff signal $\bar{s_i}$ is monotone increasing in the ratio of the conditional probability density functions $\frac{\pi(s|\alpha)}{\pi(s|\beta)}, \forall s \in [0,1]$.*

The proofs of both lemmas are given in the Appendix. It follows directly that the cutoff signal values satisfy $0 \leq \bar{s_1} \leq ... \leq \bar{s_n} < 1$. All receivers vote for A when $s \in [0, \bar{s_1}]$ and vote for B when $s \in [\bar{s_n}, 1]$. For signal realizations $s \in [\bar{s_1}, \bar{s_n}]$, the number of receivers who vote for each alternative varies. Thus under public persuasion the sender chooses a persuasion mechanism which maximizes the probability of a signal realization s such that $\bar{s_m} \leq s < 1$. Under similar lines of reasoning as Proposition 3.2.1, we have the following result:

Proposition 8. *Under public persuasion with a continuous signal realization space $S = [0,1]$, suppose the receivers adopt cutoff voting strategy described in Lemma 3. Then for any voting rule m and any common prior $p \in [0,1]$, the sender can achieve the expected payoffs on the concave closure of the set of all possible payoffs, i.e.*

$$EU^S_{PUB}(p) = \Lambda(p)$$

where $\Lambda(p)$ is defined in Section 4.2. The proof of this result is included in the Appendix.

7 Conclusion

This paper analyzes a Bayesian persuasion game with one sender persuading multiple receivers. We compare public persuasion in which both the sender's choice of mechanism and the generated signals are observed by all receivers publicly, to private persuasion in which the former remains commonly known while the latter is drawn independently and separately for each receiver. Public persuasion gives rise to common belief of all receivers. There is a unique subgame perfect equilibrium in which the sender establishes a mechanism to persuade just enough receivers to vote informatively upon a sender-favorable signal realization. Under private persuasion, in contrast, the sender no longer has full control over the receivers' posterior beliefs. By setting up a persuasion mechanism, the sender has to balance over the *information precision effect*, the *Binomial distribution effect*, and the *strategic voting effect*. There exist multiple equilibria under private persuasion, some of which involve a sufficiently large portion of receivers voting for the default option uninformatively.

Our results indicate that there is a cost for the sender to communicate in a private environment. We find that the sender is weakly better off under public persuasion. In fact, the sender can always achieve the upper bound of the set of her expected payoffs via public persuasion. In addition, we show that the distinction between public and private persuasion is not driven by the disparity in the numbers of signal realizations. The sender's welfare ranking is robust to the number of signal draws. If n pieces of evidence would be examined by a group of receivers, the sender would prefer showing all the n pieces to all receivers publicly to showing each piece to each receiver privately.

Our results also indicate that receivers' benefit from a private communication environment when the value of the pivotal voter's q_m is relatively small. It results from the receivers' strategic voting under private persuasion. On the other hand, the welfare-improving effect of private environment is limited: receivers' benefits diminish as the m-th receiver's preference bias exceeds the arithmetic mean of all others' threshold doubts. Moreover, voting fully aggregates the receivers' private information in the state where the sender and the receivers' preferences are perfectly aligned, while full information aggregation may fail in other states.

8 Appendix: Proofs and Calculations

8.1 The Implication from the Monotone Likelihood Ratio Property

Observation 1. *For any signal-generating mechanism* $\{\pi(\cdot|t)\}_{t\in T}$ *that satisfies the* Monotone Likelihood Ratio Property*, with a binary state space* $T = \{\alpha, \beta\}$*, the CDF* $\Pi(\cdot|\beta)$ first order stochastic dominates $\Pi(\cdot|\alpha)$.

Proof The MLRP implies that for every $s > s'$ and $t > t'$,

$$\frac{\pi(s|t)}{\pi(s|t')} \geq \frac{\pi(s'|t)}{\pi(s'|t')}$$

given any $s^* \in S$,

- if $s^* < s$, we have

$$\begin{aligned} \pi(s|\beta)\cdot\pi(s^*|\alpha) &\geq \pi(s^*|\beta)\cdot\pi(s|\alpha) \\ \text{so that} \int_{s\in S, s^*<s} \pi(s|\beta)\cdot\pi(s^*|\alpha)ds &\geq \int_{s\in S, s^*<s} \pi(s^*|\beta)\cdot\pi(s|\alpha)ds \\ \text{implies } \pi(s^*|\alpha)\cdot(1-\Pi(s^*|\beta)) &\geq \pi(s^*|\beta)\cdot(1-\Pi(s^*|\alpha)) \end{aligned}$$

- if $s^* \geq s$, similarly we have

$$\pi(s^*|\alpha)\cdot\Pi(s^*|\beta) \leq \pi(s^*|\beta)\cdot\Pi(s^*|\alpha)$$

Therefore we have $\forall s^* \in S$,

$$\begin{aligned} \frac{\Pi(s^*|\beta)}{\Pi(s^*|\alpha)} \leq \frac{\pi(s^*|\beta)}{\pi(s^*|\alpha)} &\leq \frac{1-\Pi(s^*|\beta)}{1-\Pi(s^*|\alpha)} \\ \text{implies } \Pi(s^*|\beta) &\leq \Pi(s^*|\alpha) \end{aligned}$$

The last inequality indicates that $\Pi(\cdot|\beta)$ FOSDs $\Pi(s|\alpha)$. We replace $\int$ with $\sum$ in this exercise when S is finite.

8.2 Proofs for Lemmas, Observations, and Propositions

8.2.1 Proof of Lemma 1

Fix a sender's persuasion mechanism $\{\pi(\cdot|t)\}_{t\in T}$. Let $x = \pi(b|\alpha)$ and $y = \pi(b|\beta)$. Assumption 2 is now equivalent to $y > x$. Receiver i's posterior belief $\mu(\beta|\cdot)$ upon observing each signal realization is

$$\begin{aligned}\mu(\beta|a) &= \frac{(1-y)(1-p)}{(1-x)p+(1-y)(1-p)} \\ \mu(\beta|b) &= \frac{y(1-p)}{xp+y(1-p)}\end{aligned}$$

First notice that $\mu(\beta|a) < \mu(\beta|b)$ under Assumption 2. As shown in Section 2.3, a receiver i votes for B if and only if $\mu(\beta|\cdot) \geq q_i$. Therefore take $\bar{q} = \mu(\beta|b) = \frac{y(1-p)}{xp+y(1-p)}$. It is obvious that $\bar{q} \in [0,1]$. For all receivers with $q_i \leq \bar{q}$, we have $q_i \leq \mu(\beta|b)$, which indicates that these receivers will vote for B on signal $s = b$. For the receivers with $q_j > \bar{q}$, we have $q_j > \mu(\beta|b)$, which means they votes for A on signal $s = b$.

Note that this lemma does not exclude the possibility that some receivers vote *uninformatively* for B after observing either $s = a$ or b signal realization. For receivers whose q_j's are sufficiently small, it is possible that $\mu(\beta|a) > q_j$. However, the following claim shows that the portion of receivers who always vote for B will not dominate the voting outcome:

Claim: Under Assumption 1 and 2, the number of receivers who vote for B regardless of the signal observation is strictly less than m.

Proof of the claim: We only need to show that $\mu(\beta|b) \geq q_m$ implies $\mu(\beta|a) < q_m$ under Assumption 1 and 2. Suppose there exists mechanism $(x,y) \in [0,1]^2$ satisfying Assumption 2 such that $\mu(\beta|b) \geq q_m$ and $\mu(\beta|a) \geq q_m$. This implies

$$\begin{aligned} & \frac{(1-p)(1-q_m)}{p \cdot q_m} \cdot y \geq x \text{ and} \\ & \frac{(1-y)(1-p)}{(1-x)p+(1-y)(1-p)} \geq q_m \\ \Rightarrow\quad & \frac{(1-p)(1-q_m)}{p \cdot q_m} \cdot (1-y) \geq 1-x \geq 1 - \frac{(1-p)(1-q_m)}{p \cdot q_m} \cdot y \end{aligned}$$

It violates Assumption 1 in which $q_j > 1-p, \forall j \in \{m, ..., n\}$. Note that when the receivers share common interests, the above derivation continue to hold when we replace q_m with a common threshold doubt q.

8.2.2 Proof of Lemma 2

Fix a sender's persuasion mechanism $\{\pi(\cdot|t)\}_{t\in T}$. Let $x = \pi(b|\alpha)$ and $y = \pi(b|\beta)$. Denote $\gamma(k,r)$ the receiver's posterior belief μ_i^β when k out of r signals are $s = b$:

$$\gamma(k,r) = \frac{(1-p)\cdot\binom{r}{k}\cdot y^k(1-y)^{r-k}}{p\cdot\binom{r}{k}\cdot x^k(1-x)^{r-k} + (1-p)\cdot\binom{r}{k}\cdot y^k(1-y)^{r-k}}$$

Assumption 2 implies that $\frac{1-x}{1-y} > \frac{x}{y}, \forall 1 \le k \le r \le n$. Therefore the following hold:

$$\begin{aligned}
\gamma(k-1,r) &< \gamma(k,r)\\
\gamma(k,r+1) &< \gamma(k,r)\\
\gamma(k,r) &< \gamma(k+1,r+1)\\
\gamma(k-1,r) &< \gamma(k,r+1)
\end{aligned}$$

Moreover, given any r independent draws, suppose receiver i knows that there are $k-1$ signals $s = b$. Then his posterior belief is $\mu_i^\beta = \gamma(k,r) \in (0,1)$ if himself observes $s = b$ and $\gamma(k-1,r) \in (0,1)$ if himself observes $s = a$.

Under private persuasion, possible voting profiles in the voting subgame can be classified into three classes. Below we examine all voting profiles in each class and exclude those violating sequential rationality.

Class 1: Voting profiles involving at least m receivers voting for B regardless of own signal observation (thereafter *uninformatively*). Under Assumption 1 and 2 **no** voting profile in this class satisfies sequential rationality. We show this by considering the following two sub-classes.

Class 1.1: At least $m+1$ receivers vote for B uninformatively.

Denote the number of receivers who vote uninformatively for B as K. Each of the rest $n-K$ receivers may vote either informatively or for A uninformatively. Since $K \ge m+1$ and m is enough to implement option B, no receiver, regardless of his voting strategy, is pivotal. As implied by the no-weakly-dominated-strategy of the receivers, a receiver still adopts the threshold-voting strategy when his vote is not pivotal. This receiver casts vote by comparing the posterior belief μ_i^β with his threshold doubt q_i. Moreover, when no one is ever pivotal, a receiver is unable to infer the distribution of others' votes, or the distribution of others' privately observations, from the situation. Thus the receiver's posterior belief is simply $\mu_i(\beta|s)$, i.e. the posterior belief given his own signal observation s.

In this subclass, there are $K \geq m+1$ receivers votes for B uninformatively. It implies $\gamma(1,1) = \mu_K(\beta|b) \geq q_K$ and $\gamma(0,1) = \mu_K(\beta|a) \geq q_K$. However, as we show above, $\gamma(1,1) \geq q_m$ implies $\gamma(0,1) < q_m$. Note that $q_K \geq q_m$. A contradiction. Therefore this subclass of voting profiles violates sequential rationality.

Class 1.2: Exact m receivers vote for B uninformatively. Each of the rest $n-m$ receivers either votes informatively or votes for A uninformatively.

For each of the m receivers who votes for B uninformatively, being pivotal indicates that $m-1$ votes for B and $n-m$ votes for A are among all others' votes. Besides himself, there are $m-1$ receivers voting for B uninformatively. Therefore, one observing $s=a$ and being pivotal implies that there is no $s=b$ signal realization among $n-m+1$ (if all the other $n-m$ receivers vote informatively) or less (if less than $n-m$ other receivers vote informatively) realizations that a receiver can observe or infer in total. And observing $s=b$ and being pivotal indicates that there is one $s=b$ signal realization among $n-m+1$ or less realizations in total. For these receivers, voting uninformatively for B implies:

$$\gamma(0,\chi) \geq q_j, \gamma(1,\chi) \geq q_j \text{ where } 1 \leq \chi \leq n-m+1$$

For the rest $n-m$ receivers who may either vote informatively or vote for A uninformatively, none is ever pivotal since the m votes for B are enough to implement B regardlessly. As implied by the no-weakly-dominated-strategy criterion, each of these receivers' posterior belief is simply $\mu_i(\beta|s)$, given one's own signal observation s. And the receivers adopt the threshold-voting strategy. These receivers' posterior beliefs satisfy:

$$\gamma(0,1) < q_i, \gamma(1,1) \geq q_i \text{ receiver } i \text{ votes informatively}$$
$$\gamma(0,1) < q_k, \gamma(1,1) < q_k, \text{ receiver } k \text{ votes for } A \text{ uninformatively}$$

Assumption 2 implies $\gamma(0,\chi) < \gamma(0,1), 1 \leq \chi \leq n-m+1$. Assumption 1 implies $\gamma(0,1) < q_m$ when $\gamma(1,1) \geq q_m$. But for this subclass to hold we need at least $q_m \leq \gamma(0,\chi)$. A contradiction.

In summary, under Assumption 1 and 2 all voting profiles in *Class 1* violate sequential rationality. Thus we prove the first part of Lemma 2, i.e. the number of receivers who vote for B regardless of signal observation is strictly less than m. It is also worthy noting that when the receivers share common interests, the above derivation holds when we replace the heterogeneous q_i's with a common threshold doubt q.

Class 2: Voting profiles involving at least $n-m+1$ receivers to vote for A uninformatively. We consider the following two subclasses.

Class 2.1: At least $n-m+2$ receivers vote for A uninformatively.

Denote the number of receivers who vote uninformatively for A as J. Each of the rest $n-J$ receivers may vote either informatively or for B uninformatively. Since $J \geq n-m+2$ and $n-m+1$ is enough to keep the default option A, no receiver, regardless of his voting strategy, is pivotal. Therefore each receiver casts his vote by comparing his threshold doubt q_i with the posterior belief $\mu_i(\beta|s)$ given his own signal observation s. The J receivers voting for A uninformatively implies

$$\gamma(0,1) < q_k \text{ and } \gamma(1,1) < q_k \text{ for all receiver } k$$

For the rest $n-J$ receivers, threshold voting implies

$$\begin{aligned}
&\gamma(0,1) < q_i, \gamma(1,1) \geq q_i \text{ receiver } i \text{ votes informatively} \\
&\gamma(0,1) \geq q_j, \gamma(1,1) \geq q_j, \text{ receiver } j \text{ votes for } B \text{ uninformatively}
\end{aligned}$$

Note that *Class 2.1* includes two voting profiles in which *all* receivers vote regardless of own signal observations: one involves every receiver voting for A uninformatively, and the other involves J receivers voting for A, the rest $n-J$ receivers voting for B uninformatively.

When the receivers share common interests, the threshold doubts q_i's reduces to one parameter q. Since $\gamma(1,1) \geq q$ implies $\gamma(0,1) < q$, it must be either $q \in [\gamma(0,1), \gamma(1,1)]$, or $q < \gamma(0,1)$. Only the latter holds for *Class 2.1.* Thus *all* receivers voting uninformatively for A, namely, $J = n$, is the only voting profile satisfying sequential rationality for common-interest receivers.

Class 2.2: Exact $n-m+1$ receivers vote for A uninformatively. Each of the rest m receivers either votes informatively or votes for B uninformatively.

For each of the $n-m+1$ receivers voting for A uninformatively, being pivotal indicates that $m-1$ votes for B and $n-m$ votes for A are among all other votes. Besides himself, there are $n-m$ receivers voting for A uninformatively. Therefore, one observing $s=a$ and being pivotal implies that there are $m-1$ signal realizations as $s=b$ among m (if all the other m receivers vote informatively) or less (if less than m other receivers vote informatively) realizations in total. And observing $s = b$ and being pivotal indicates that all signal realizations are $s = b$. For these receivers, voting uninformatively for A implies:

$$\gamma(\chi - 1, \chi) < q_k, \gamma(\chi, \chi) < q_k \text{ where } 1 \leq \chi \leq m$$

For the rest $m-1$ receivers who may either vote informatively or vote for B uninformatively, none is ever pivotal since $n-m+1$ votes are enough to keep the default option A. Thus each receiver's posterior belief is simply $\mu_i(\beta|s)$ given his own signal observation s. Threshold voting strategy implies

$$\gamma(0,1) < q_i, \gamma(1,1) \geq q_i \text{ receiver } i \text{ votes informatively}$$
$$\gamma(0,1) \geq q_j, \gamma(1,1) \geq q_j, \text{ receiver } j \text{ votes for } B \text{ uninformatively}$$

Now we consider receivers sharing common interests with threshold doubt q. For $n - m + 1$ receivers voting for A regardlessly, we need $\gamma(\chi, \chi) < q, 1 \leq \chi \leq m$. For the rest of the receivers, $q < \gamma(0,1)$ implies voting for B uninformatively, or $q \in [\gamma(0,1), \gamma(1,1)]$ implies informative voting. But $\gamma(1,1) < \gamma(\chi, \chi)$. A contradiction. Thus no voting profile in *Class 2.2* satisfies sequential rationality if receivers have common interests.

Among all sequentially rational voting profiles in *Class 2*, suppose the receivers' preferences are sufficiently close. For all possible mechanisms that satisfy MLRP, there exists a threshold doubt value $q \in [1-p, 1]$ and $\epsilon > 0$ such that when all receivers' threshold doubts falls between the interval $q_i \in [q-\epsilon, q+\epsilon], \forall i \in \{1, ..., N\}$, the only voting profile that satisfies sequential rationality in *Class 2* involves *all* receivers voting for A regardless of signal observation.

Class 3: All voting profiles that do not belong to *Class 1* or *Class 2*. This class includes voting profiles in which $m - \ell$ receivers vote uninformatively for B, $n - m - k$ receivers vote uninformatively for A, and $\ell + k$ receivers vote informatively, for $1 \leq \ell \leq m$ and $0 \leq k \leq n - m$.

For each of the $m - \ell$ receivers voting for B uninformatively, besides himself there are $m - \ell - 1$ receivers voting for B uninformatively. Therefore, one observing $s = a$ and being pivotal implies that there are ℓ realizations as $s = b$ among the $1 + k + \ell$ informative votes. And observing $s = b$ and being pivotal indicates that there are $\ell + 1$ realizations as $s = b$ among the $1 + k + \ell$ informative votes. For these receivers, voting uninformatively for B

implies:

$$\gamma(\ell, k+\ell+1) \geq q_j, \gamma(\ell+1, k+\ell+1) \geq q_j \tag{3}$$

For each of the $n-m-k$ receivers voting for A uninformatively, besides himself, there are $n-m-k-1$ receivers voting for A uninformatively. Therefore, one observing $s=a$ and being pivotal implies that there are $\ell-1$ realizations as $s=b$ among all $1+k+\ell$ informative votes. And observing $s=b$ and being pivotal indicates that there are ℓ realizations as $s=b$ among all $1+k+\ell$ informative votes. For these receivers, voting uninformatively for A implies:

$$\gamma(\ell-1, k+\ell+1) < q_k, \gamma(\ell, k+\ell+1) < q_k \tag{4}$$

For the $\ell+k$ receivers who vote informatively, being pivotal indicates that $m-1$ votes for B and $n-m$ votes for A are among all other votes. Since there are $m-\ell$ uninformative votes for B and $n-m-k$ uninformative votes for A, the total number of informative signal realizations that one can infer from pivotality is $\ell+k$. Therefore, one observing $s=a$ and being pivotal implies that there are $\ell-1$ realizations as $s=b$ among all informative votes; observing $s=b$ and being pivotal indicates that there are ℓ realizations as $s=b$ among all informative votes. For these receivers, voting informatively according to his *own* signal observation implies:

$$\gamma(\ell-1, k+\ell) < q_i, \gamma(\ell, k+\ell) \geq q_i$$

It is worthy noting that for each given ℓ and k and some persuasion mechanism (x, y), there may exist more than one voting profiles that satisfying sequential rationality. We classify them as *symmetric* and *asymmetric* voting profiles:

Definition 7. *A voting profile is* ***symmetric*** *if for any two receivers j, i with threshold doubts $q_j < q_i$, (1) receiver j voting informatively implies receiver i voting informatively, or uninformatively for A; and (2) receiver j uninformatively voting for A implies receiver i also uninformatively voting for A. Any voting profile that does not satisfy the above conditions is called* ***asymmetric****.*

For any given ℓ and k, consider all possible voting profiles. By Assumption 2, we have $\gamma(\ell-1, k+\ell+1) < \gamma(\ell-1, k+\ell) < \gamma(\ell, k+\ell+1) < \gamma(\ell, k+\ell) < \gamma(\ell+1, k+\ell+1)$. Receivers with threshold doubts $q_j < \gamma(\ell, k+\ell+1)$ vote uninformatively for B, with $q_i \in [\gamma(\ell-1, k+\ell), \gamma(\ell, k+\ell)]$ vote informatively according to one's signal observations, and with $q_k > \gamma(\ell, k+\ell+1)$ vote uninformatively for A. Thus a **symmetric** voting profile involves the $m-\ell$ receivers with the smallest threshold doubts $q_1, ..., q_{m-\ell}$ voting uninformatively for B, the $k+\ell$ receivers in the middle with $q_{m-\ell+1}, ..., q_{m+k}$ voting informatively, and the

$n-m-k$ receivers with the largest threshold doubts $q_{m+k+1}, ..., q_n$ voting uninformatively for A.

In addition, there exist some **asymmetric** voting profiles because $\gamma(\ell-1, k+\ell) < \gamma(\ell, k+\ell+1) < \gamma(\ell, k+\ell)$. Consider two receivers $q_j < q_i$ with both threshold doubts between $\gamma(\ell-1, k+\ell)$ and $\gamma(\ell, k+\ell+1)$. Receiver j may vote informatively while receiver i votes for B uninformatively. Similarly, if any two receivers with threshold doubts $q_j < q_i$ between $\gamma(\ell, k+\ell+1)$ and $\gamma(\ell, k+\ell)$, receiver j may vote for A uninformatively while i votes informatively. All such voting profiles satisfy sequential rationality.

Next we show the second part of Lemma 2 by considering multiple sequentially rational voting profiles induced by one persuasion mechanism $(\hat{x}, \hat{y})$ across the *classes* specified above. Consider the pair $\hat{x} = \sqrt[m]{(1-p)(1-q_m)/p \cdot q_m}, \hat{y} = 1$. It is easy to check that it can induce a voting profile that m receivers vote informatively, and $n-m$ receivers vote uninformatively for A. This is because $q_1, ...q_m \in [\hat{\gamma}(m-1, m), \hat{\gamma}(m, m)]$ and $q_{m+1} > \hat{\gamma}(m, m+1)$ given $(\hat{x}, \hat{y})$.

In addition, this mechanism $(\hat{x}, \hat{y})$ can also induce the voting profiles of *Class 2*. Note that $m \geq 1$ implies

$$\begin{aligned} & \frac{(1-p)(1-q_m)}{p \cdot q_m} \cdot \hat{y} < \hat{x} \\ \Rightarrow \quad & \frac{(1-p) \cdot \hat{y}}{p \cdot \hat{x} + (1-p) \cdot \hat{y}} < q_m \\ \Rightarrow \quad & \hat{\gamma}(1, 1) < q_m \end{aligned}$$

Similarly, $\hat{\gamma}(\chi, \chi) < q_m, \forall \chi \leq m-1$. The conditions indicate that the number of receivers who vote uninformatively for A regardless of signal observation is at least $n-m+1$, i.e. voting profiles in either *Class 2.1* or *Class 2.2*.

8.3 Proof of Corollary 1

We show that no voting profile in which uninformative-vote-for-A receiver(s) and uninformative-vote-for-B receiver(s) exist at the same time when receivers' preferences are sufficiently close.

Consider receivers share a common threshold doubt q. For all $1 \leq \ell \leq m-1$, and all $0 \leq k \leq n-m-1$, it is easy to check that the first equation in condition 3 and the second equation in condition 4 cannot hold simultaneously. Therefore, for common-interest receivers, the only possible voting profiles in *Class 3* are either $\ell = m$, i.e. $n-m-k$ receivers vote for A uninformatively, and $k+m$ receivers vote informatively ($0 \leq k \leq n-m$); or $k = n-m$, i.e. $m-\ell$ receivers vote for B uninformatively, and $n-m+\ell$ receivers vote informatively ($1 \leq \ell \leq m$). Any voting profile that involves at least one receiver voting uninformatively for A and at least one receivers voting uninformatively for B at the same time is excluded.

Now suppose the receivers' preferences are sufficiently close. For every possible mechanisms that satisfy MLRP, there exists a threshold doubt value $q \in [1-p, 1]$ and $\epsilon > 0$ such that when all receivers' threshold doubts falls between the interval $q_i \in [q-\epsilon, q+\epsilon], \forall i \in \{1, ..., N\}$, any sequentially rational voting profile should not include the uninformative-for-A voters and the uninformative-for-B voters at the same time.

8.3.1 Proof of Proposition 1

Let $x = \pi(b|\alpha)$ and $y = \pi(b|\beta)$ denote the sender's choice of the conditional probabilities. Under public persuasion, all receivers observe the same signal realization s. As shown in Lemma 1, no receiver votes for B upon observing signal realization $s = a$. Upon observing $s = b$, if at least m receivers vote for B, the final decision is B. This requires the receivers' posterior belief upon observing $s = b$ to satisfy:

$$\mu(\beta|b) = \frac{y(1-p)}{xp + y(1-p)} \geq q_m$$

since $q_m \geq ... \geq q_1$, this inequality implies that receivers $\{R_1, ..., R_m\}$ will vote for B upon observing $s = b$. So the sender's problem is:

$$\begin{aligned}
&\max_{x,y \in [0,1]^2} g(x,y) = p \cdot x + (1-p) \cdot y \\
&\text{subject to } \frac{y(1-p)}{xp + y(1-p)} \geq q_m \\
&0 \leq x \leq 1, 0 \leq y \leq 1
\end{aligned}$$

where the objective function is the sum of the probabilities that signal $s = b$ is generated in each state.

First notice that the $g(x, y)$ is continuous in (x, y). The constraint set, $\{(x, y) \in [0, 1]^2 | \frac{y(1-p)}{xp+y(1-p)} \geq q_m\}$ is closed and bounded in $\mathbb{R}^2$; thus it is compact. By

Weierstrass's extreme value theorem, the maximum of the objective function is attainable. We form the Lagrangian function

$$L(x, y, \lambda) = g(x, y) - \lambda(-\frac{y(1-p)}{xp + y(1-p)} + q_m)$$

the Kuhn-Tucker condition yields the following equations and inequalities:

$$p + \frac{\lambda p(1-p)y}{(px + (1-p)y)^2} = 0$$
$$(1-p) + \lambda \frac{p(1-p)x}{(px + (1-p)y)^2} = 0$$
$$\lambda(-\frac{y(1-p)}{xp + y(1-p)} + q_m) = 0$$
$$\lambda \geq 0$$
$$\frac{y(1-p)}{xp + y(1-p)} \geq q_m$$

solving the equations and inequalities yields $x^* = \frac{(1-q_m)\cdot(1-p)}{q_m \cdot p}$, $y^* = 1$. In fact, the constraint is binding, i.e. $\mu(\beta|b) = q_m$.

Now we show that there are exactly m receivers who vote for B upon observing $s = b$. Suppose there are only $m-1$ receivers who do so. In this case the voting outcome is always $v = A$ in each state, which yields the sender only $u^S(A, t)$. But the sender's optimal mechanism yields $u^S(A, t) + (u^S(B, t) - u^S(A, t)) \cdot g(x^*, y^*) > u^S(A, t)$. Now suppose instead there are $m+1$ receivers who vote for B upon observing $s = b$, which changes the constraint to $\mu(\beta|b) \geq q_{m+1}$. Solving the sender's problem we get $x' = \frac{(1-q_{m+1})\cdot(1-p)}{q_{m+1} \cdot p}$, $y' = 1$. Since $q_{m+1} \geq q_m$, $g(x', y') \leq g(x^*, y^*)$. Therefore, in equilibrium there are exact m receivers who vote for B upon observing $s = b$, i.e. $\bar{q} = \mu(\beta|b) = q_m$.

8.3.2 Proof of Proposition 2 and 3

Let $x = \pi(b|\alpha)$ and $y = \pi(b|\beta)$ denote the sender's choice of the conditional probabilities. Since $EU^S = u^S(A, t) + (u^S(B, t) - u^S(A, t)) \cdot \text{Prob}(v = B), \forall t \in T$, to maximize the sender's expected payoffs from the persuasion mechanism, we only need to calculate the conditional probabilities that maximizes $\text{Prob}(v = B)$. Under public persuasion, sender's optimal mechanism satisfies $x^*_{\text{PUB}} = \frac{(1-q_m)\cdot(1-p)}{q_m \cdot p}$, $y^*_{\text{PUB}} = 1$. Then we have:

$$\text{Prob}_{\text{PUB}}(v = B) = p \cdot \frac{(1-q_m)\cdot(1-p)}{q_m \cdot p} + (1-p) \cdot 1$$

Under private persuasion, first observe that all voting profiles in *Class 2* yield the sender an expected payoff of 0. Next we consider voting profiles in *Class 3*, i.e. $m-\ell$ receivers vote uninformatively for B, $n-m-k$ receivers vote uninformatively for A, and $\ell+k$ receivers vote informatively ($1 \leq \ell \leq m$ and $0 \leq k \leq n-m$), which may may yield the sender a strictly positive expected payoff. For symmetric voting profiles induced in the voting subgame, the sender chooses $(x,y) \in [0,1]^2$ to maximize the probability of at least ℓ receivers' observing $s=b$ of all $\ell+k$ receivers who vote informatively subject to the following constraints:

$$
\begin{aligned}
\max_{x,y} g(x,y) \quad &= \quad p \cdot [\sum_{i=\ell}^{\ell+k} \binom{\ell+k}{i} \cdot x^i \cdot (1-x)^{\ell+k-i}] + (1-p) \cdot [\sum_{i=\ell}^{\ell+k} \binom{\ell+k}{i} \cdot y^i \cdot (1-y)^{\ell+k-i}] \\
\text{subject to} \quad & q_{m-\ell} \leq \gamma(\ell, 1+k+\ell) < q_{m+k+1} \\
& \gamma(\ell-1, k+\ell) < q_{m-\ell+1} \\
& q_{m+k} \leq \gamma(\ell, k+\ell) \\
& 0 \leq x \leq 1, 0 \leq y \leq 1
\end{aligned}
$$

Notice that the $g(x,y)$ is continuous and monotone increasing in (x,y). The constraint set, $\{(x,y) \in [0,1]^2 | q_{m-\ell} \leq \gamma(\ell, 1+k+\ell) \leq q_{m+k+1}, \gamma(\ell-1, k+\ell) \leq q_{m-\ell+1} \leq q_{m+k} \leq \gamma(\ell, k+\ell)\}$ is closed and bounded in $\mathbb{R}^2$; thus it is compact. By *Weierstrass's extreme value theorem*, the maximum of the objective function is attainable. We form the Lagrangian function:

$$
\begin{aligned}
& L(x,y;\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_x,\lambda_y,\nu_x,\nu_y) = g(x,y) - \lambda_1 \cdot (-\gamma(\ell, 1+k+\ell) + q_{m-\ell}) \\
& -\lambda_2 \cdot (\gamma(\ell, 1+k+\ell) - q_{m+k+1}) - \lambda_3 \cdot (-\gamma(\ell, k+\ell) + q_{m+k}) \\
& -\lambda_4 \cdot (\gamma(\ell-1, k+\ell) - q_{m-\ell+1}) - \lambda_x \cdot (x-1) - \lambda_y \cdot (y-1) + \nu_x \cdot x + \nu_y \cdot y
\end{aligned}
$$

The corresponding Kuhn-Tucker conditions are:

$$
\begin{aligned}
& \frac{\partial g(x,y)}{\partial x} + (\lambda_1 - \lambda_2) \cdot \frac{\partial \gamma(\ell, 1+k+\ell)}{\partial x} + \lambda_3 \cdot \frac{\partial \gamma(\ell, k+\ell)}{\partial x} - \lambda_4 \cdot \frac{\partial \gamma(\ell-1, k+\ell)}{\partial x} - \lambda_x + \nu_x = 0 \\
& \frac{\partial g(x,y)}{\partial y} + (\lambda_1 - \lambda_2) \cdot \frac{\partial \gamma(\ell, 1+k+\ell)}{\partial y} + \lambda_3 \cdot \frac{\partial \gamma(\ell, k+\ell)}{\partial y} - \lambda_4 \cdot \frac{\partial \gamma(\ell-1, k+\ell)}{\partial y} - \lambda_y + \nu_y = 0 \\
& \lambda_1 \cdot (-\gamma(\ell, 1+k+\ell) + q_{m-\ell}) = 0 \\
& \lambda_2 \cdot (\gamma(\ell, 1+k+\ell) - q_{m+k+1}) = 0 \\
& \lambda_3 \cdot (-\gamma(\ell, k+\ell) + q_{m+k}) = 0 \\
& \lambda_4 \cdot (\gamma(\ell-1, k+\ell) - q_{m-\ell+1}) = 0
\end{aligned}
$$

$$
\begin{aligned}
&-\gamma(\ell, 1+k+\ell) \le -q_{m-\ell} \\
&\gamma(\ell, 1+k+\ell) \le q_{m+k+1} \\
&-\gamma(\ell, k+\ell) \le -q_{m+k} \\
&\gamma(\ell-1, k+\ell) \le q_{m-\ell+1} \\
&\lambda_x \cdot (x-1) = 0, \lambda_y \cdot (y-1) = 0, \nu_x \cdot x = 0, \nu_y \cdot y = 0 \\
&0 \le x \le 1, 0 \le y \le 1 \\
&\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_x, \lambda_y, \nu_x, \nu_y \ge 0
\end{aligned}
$$

Observe that Assumption 2 and the receivers' heterogeneous threshold doubts implies:

- The constraint $\gamma(\ell, k+\ell) \ge q_{m+k}$ is binding $\Rightarrow \gamma(\ell, 1+k+\ell) < q_{m+k+1} \Rightarrow \lambda_2 = 0$.
- The constraint $\gamma(\ell, 1+k+\ell) \ge q_{m-\ell}$ is binding $\Rightarrow \gamma(\ell-1, k+\ell) < q_{m-\ell+1} \Rightarrow \lambda_4 = 0$.

First the constraint $q_{m+k} \le \gamma(\ell, k+\ell)$ is binding. By Assumption 2 we have the following:

$$
\begin{aligned}
q_{m+k} &\le \gamma(\ell, k+\ell) \\
&= \frac{(1-p)\cdot\binom{k+\ell}{\ell}\cdot y^\ell\cdot(1-y)^k}{p\cdot\binom{k+\ell}{\ell}\cdot x^\ell\cdot(1-x)^k + (1-p)\cdot\binom{k+\ell}{\ell}\cdot y^\ell\cdot(1-y)^k} \\
&< \frac{(1-p)\cdot\binom{k+\ell}{\ell+1}\cdot y^{\ell+1}\cdot(1-y)^{k-1}}{p\cdot\binom{k+\ell}{\ell+1}\cdot x^{\ell+1}\cdot(1-x)^{k-1} + (1-p)\cdot\binom{k+\ell}{\ell+1}\cdot y^{\ell+1}\cdot(1-y)^{k-1}} \\
&\dots \\
&< \frac{(1-p)\cdot\binom{k+\ell}{k+\ell}\cdot y^{k+\ell}}{p\cdot\binom{k+\ell}{k+\ell}\cdot x^{k+\ell} + (1-p)\cdot\binom{k+\ell}{k+\ell}\cdot y^{k+\ell}} = \gamma(k+\ell, k+\ell)
\end{aligned}
$$

Re-arrange terms, we have

$$
\begin{aligned}
x^\ell\cdot(1-x)^k &\le \frac{(1-p)(1-q_{m+k})}{p\cdot q_{m+k}}\cdot y^\ell\cdot(1-y)^k \\
x^{\ell+1}\cdot(1-x)^{k-1} &< \frac{(1-p)(1-q_{m+k})}{p\cdot q_{m+k}}\cdot y^{\ell+1}\cdot(1-y)^{k-1} \\
&\dots \\
x^{\ell+k} &< \frac{(1-p)(1-q_{m+k})}{p\cdot q_{m+k}}\cdot y^{\ell+k}
\end{aligned}
$$

Note that all except the first one are strict inequalities. Insert the above inequalities into the sender's problem, and note that $\sum_{i=\ell}^{\ell+k}\binom{\ell+k}{i}\cdot y^i\cdot(1-y)^{\ell+k-i}<1$, we have

$$\begin{aligned} Eg_{\text{PRI}}(x,y) &< p\cdot[\sum_{i=\ell}^{\ell+k}\binom{\ell+k}{i}\cdot\frac{(1-p)(1-q_{m+k})}{p\cdot q_{m+k}}\cdot y^i\cdot(1-y)^{k+\ell-i}] \\ &+ (1-p)\cdot[\sum_{i=\ell}^{\ell+k}\binom{\ell+k}{i}\cdot y^i\cdot(1-y)^{\ell+k-i}] \\ &= \frac{1-p}{q_{m+k}}\cdot[\sum_{i=\ell}^{\ell+k}\binom{\ell+k}{i}\cdot y^i\cdot(1-y)^{\ell+k-i}] < \frac{1-p}{q_{m+k}} \end{aligned}$$

Thus the upper bound of function $Eg_{\text{PRI}}(x,y)$ is smaller than $(1-p)/q_m$, the value of $Eg^*_{\text{PUB}}(x,y)$ under public persuasion. Notice that when $k=0$, i.e. $m-\ell$ receivers vote uninformatively for B, $n-m$ receivers vote uninformatively for A, and ℓ receivers vote informatively ($1\le\ell\le m$), the binding constraint becomes $q_m\le\gamma(\ell,\ell)$ and the objective function reduces to:

$$Eg_{\text{PRI}}(x,y)=p\cdot\binom{\ell}{\ell}\cdot x^\ell+(1-p)\cdot\binom{\ell}{\ell}\cdot y^\ell$$

Meanwhile, the other binding constraint becomes $\gamma(\ell,1+\ell)\ge q_{m-\ell},\forall\ell<m$. Since $((1-p)\cdot(1-q_m))/(p\cdot q_m)<1$, from the binding constraints we have:

$$\begin{aligned} & \gamma(\ell,1+\ell)=q_{m-\ell},\gamma(\ell,\ell)=q_m \\ \Rightarrow\quad & x=\sqrt[\ell]{\frac{(1-p)(1-q_m)}{p\cdot q_m}}\cdot y \text{ and } y=\frac{\frac{q_m\cdot(1-q_{m-\ell})}{(1-q_m)\cdot q_{m-\ell}}-1}{\frac{q_m\cdot(1-q_{m-\ell})}{(1-q_m)\cdot q_{m-\ell}}-\sqrt[\ell]{\frac{(1-p)\cdot(1-q_m)}{p\cdot q_m}}}<1 \\ \Rightarrow\quad & Eg_{\text{PRI}}(x,y)=\frac{1-p}{q_m}\cdot\binom{\ell}{\ell}\cdot y^\ell<\frac{1-p}{q_m},\forall\ell<m \end{aligned}$$

As the number of informative-voting receivers, ℓ, increases, $\sqrt[\ell]{\frac{(1-p)\cdot(1-q_m)}{p\cdot q_m}}$ increases, so the value of y and $Eg_{\text{PRI}}(x,y)$ increases monotonically. This is true for all $\ell<m$. When ℓ equals m, i.e. $n-m$ receivers vote uninformatively for A, and m receivers vote informatively, the set of constraints reduces to:

$$\gamma(m-1,m)<q_1,\gamma(m,m)\ge q_m,\text{ and }\gamma(m,m+1)<q_{m+1}$$

It is easy to check that only the middle one is binding. The objective function maximizes at $y^* = 1$:

$$
\begin{aligned}
Eg_{\text{PRI}}(x,y) &= p \cdot \binom{m}{m} \cdot \left(\sqrt[m]{\frac{(1-p)\cdot(1-q_m)}{p\cdot q_m}} \cdot y\right)^m + (1-p)\cdot\binom{m}{m}\cdot y^m \\
&= \frac{1-p}{q_m}\cdot y^m \leq \frac{1-p}{q_m} = Eg^*_{\text{PUB}}(x,y)
\end{aligned}
$$

Thus there exists one subgame perfect equilibrium in which the sender chooses a persuasion mechanism $x^* = \sqrt[m]{\frac{(1-p)\cdot(1-q_m)}{p\cdot q_m}}, y^* = 1$ to induce a symmetric voting profile in which m receivers with the smallest threshold doubts vote informatively, while the rest $n-m$ receivers vote uninformatively for A. In this equilibrium, the sender's objective function achieves the upper bound $\frac{1-p}{q_m}$, which equals the value of $Eg^*_{\text{PUB}}(x,y)$.

However, as shown in the last part of proof for Lemma 2, the mechanism (x^*, y^*) also induces voting profiles of *Class 2*. This is because for any $m \geq 1$, the posterior beliefs satisfy $\gamma^*(\chi,\chi) < \gamma^*(1,1) < q_m, \forall \chi \leq m-1$. Thus there are at least $n-m+1$ receivers who vote uninformatively for A, which yields the sender an expected payoff 0.

Admittedly, the sender could choose a mechanism $(\tilde{x},\tilde{y})$ to induce only the above-specified symmetric voting profile of *Class 3* but not any of the voting profiles of *Class 2*:

$$
\begin{aligned}
(\tilde{x},\tilde{y}) &\notin \{(x,y)\in[0,1]^2 | \gamma(m-1,m) < q_1, \gamma(m,m) \geq q_m, \gamma(m,m+1) < q_{m+1}\} \\
&\bigcap \{(x,y)\in[0,1]^2 | \gamma(\chi,\chi) < \gamma(1,1) < q_m, \forall \chi \leq m-1\}
\end{aligned}
$$

But then we have $\tilde{x}^m < \frac{(1-p)\cdot(1-q_m)}{p\cdot q_m}\cdot\tilde{y}^m$, which implies $Eg_{\text{PRI}}(x,y) < \frac{1-p}{q_m} = Eg^*_{\text{PUB}}(x,y)$. The mechanism that induces the specified voting profile no longer generates the optimal expected payoff for the sender.

So far the analysis includes the *symmetric informative-voting profiles* in *Class 3* induced by sender's persuasion mechanism. As we have shown in the proof of Lemma 2, there also exists persuasion mechanisms $(x,y)\in[0,1]^2$ that may induce receivers' asymmetric voting behavior in *Class 3*. Next, we shall demonstrate that, inducing an asymmetric voting profile lowers the sender's expected payoff: given each ℓ and k, whenever there is a mechanism that induces an asymmetric voting profile, there exists another persuasion mechanism that induces a symmetric voting profile which yields the sender strictly higher expected payoff.

Lemma 5. *Fix ℓ and k. The mechanism that induces a symmetric voting profile yields the sender strictly higher expected payoff than a mechanism that induces any asymmetric* *profiles.*

Proof of Lemma 5: For fixed ℓ and k, observe that the sender's objective function does not change no matter the induced voting profile being symmetric or asymmetric. Thus we only need to examine the constraint set which is derived from the receivers' voting behavior. First consider the receivers whose threshold doubts fall in the interval $[\gamma(\ell-1, k+\ell), \gamma(\ell, 1+k+\ell)]$. According to the proof of Lemma 2, each of these receivers may either vote uninformatively for B or vote informatively, as long as the value ℓ and k remain fixed. We shall only consider asymmetric voting profiles that change the constraint set of sender's problem. Without loss of generality, consider two receivers with threshold doubts $q_{m-\ell}$ and q_i such that $\gamma(\ell-1, k+\ell) < q_{m-\ell} < q_i < \gamma(\ell, 1+k+\ell)$. Consider an asymmetric voting profile in which the receiver with $q_{m-\ell}$ votes informatively, while the receiver with q_i votes for B uninformatively. The constraint set changes to:

$$q_i \leq \gamma(\ell, 1+k+\ell) < q_{m+k+1}, \gamma(\ell-1, k+\ell) < q_{m-\ell}, \text{ and } q_{m+k} \leq \gamma(\ell, k+\ell)$$

where the two binding constraints are $q_i \leq \gamma(\ell, 1+k+\ell)$ and $q_{m+k} \leq \gamma(\ell, k+\ell)$. Since $q_{m-\ell} < q_i$, it is easy to check that the maximizers here $x^*_{\text{asy}} < x^*_{\text{pri}}$ and $y^*_{\text{asy}} < y^*_{\text{pri}}$, where x^*_{pri} and y^*_{pri} are the above-derived maximizers of the symmetric voting profile for each ℓ and k. Thus we have $Eg_{\text{asy}}(x, y) < Eg_{\text{pri}}(x, y)$.

Second we consider the receivers whose threshold doubts fall in the interval $[\gamma(\ell, 1+k+\ell), \gamma(\ell, k+\ell)]$. Each of these receivers may either vote uninformatively for A or vote informatively. Without loss of generality, consider two receivers with threshold doubts q_{m+k+1} and q_j such that $\gamma(\ell, 1+k+\ell) < q_j < q_{m+k+1} < \gamma(\ell, k+\ell)$. Consider an asymmetric voting profile in which the receiver with q_{m+k+1} votes informatively, while the receiver with q_j votes for A uninformatively. The constraint set changes to:

$$q_{m-\ell} \leq \gamma(\ell, 1+k+\ell) < q_j, \gamma(\ell-1, k+\ell) < q_{m-\ell-1}, \text{ and } q_{m+k+1} \leq \gamma(\ell, k+\ell)$$

where the two binding constraints are $q_{m-\ell} \leq \gamma(\ell, 1+k+\ell)$ and $q_{m+k+1} \leq \gamma(\ell, k+\ell)$. Since $q_{m+k+1} \geq q_{m+k}$, it is easy to check that $Eg_{\text{asy}}(x, y) < \frac{1-p}{q_{m+k+1}} \cdot [\sum_{i=\ell}^{\ell+k} \binom{\ell+k}{i} \cdot y^i \cdot (1-y)^{\ell+k-i}] \leq Eg_{\text{pri}}(x, y)$.

8.3.3 Proof of Proposition 4

Let $x = \pi(b|\alpha)$ and $y = \pi(b|\beta)$ denote the sender's choice of the conditional probabilities. All receivers observe the same n public signal realizations; thus they all have the same posterior belief. Denote $\gamma(\ell, n)$ the posterior belief μ^β when there are ℓ signals $s = b$ out of all n signals. A receiver i with threshold doubt q_i will vote for B when there are more than ℓ favorable signals ($s = b$) out of all n signal realizations if and only if $\gamma(\ell - 1, n) < q_i < \gamma(\ell, n)$. The sender chooses the mechanism such that exact m receivers are convinced to vote for B when there are more than ℓ favorable signals, i.e. $\gamma(\ell, n) \leq q_m \leq \gamma(\ell + 1, n)$. Had the sender attempted to convince more than m receivers to vote informatively, e.g. $\gamma(\ell, n) \leq q_{m+j} \leq \gamma(\ell + 1, n), j = 1, 2, ...,$ x would decrease and the sender's objective function, Prob$(v = B)$, would decline, too. The sender's problem is:

$$\begin{aligned}
&\max_{x,y} g(x, y) = p \cdot \sum_{j=\ell}^{n} x^j (1 - x)^{n-j} + (1 - p) \cdot \sum_{j=\ell}^{n} y^j (1 - y)^{n-j} \\
&\text{subject to } \gamma(\ell, n) \leq q_m \leq \gamma(\ell + 1, n) \\
&0 \leq x \leq 1, 0 \leq y \leq 1
\end{aligned}$$

First notice that the $g(x, y)$ is continuous in (x, y). The constraint set, $\{(x, y) \in [0, 1]^2 | \gamma(\ell, n) \leq q_m \leq \gamma(\ell + 1, n)\}$ is closed and bounded in $\mathbf{R}^2$; thus it is compact. By *Weierstrass's extreme value theorem*, the maximum of the objective function is attainable. We form the Lagrangian function:

$$\begin{aligned}
&L(x, y; \lambda_1, \lambda_2, \lambda_x, \lambda_y, \nu_x, \nu_y) = g(x, y) \\
&-\lambda_1 \cdot (q_m \cdot p \cdot x^\ell (1 - x)^{n-\ell} - (1 - q_m)(1 - p)y^\ell (1 - y)^{n-\ell} - 0) \\
&-\lambda_2 \cdot (-q_m \cdot p \cdot x^{\ell-1} (1 - x)^{n-\ell+1} + (1 - q_m)(1 - p)y^{\ell-1} (1 - y)^{n-\ell+1} - 0) \\
&-\lambda_x \cdot (x - 1) - \lambda_y \cdot (y - 1) + \nu_x \cdot x + \nu_y \cdot y
\end{aligned}$$

with $p \in [0,1]; l, n \in \mathbf{N}, 1-p \leq q_m \leq 1, q'(\cdot) > 0$. The Kuhn-Tucker condition for the above constrained maximization problem yields the following equations and inequalities.

$$\begin{aligned}
&\frac{\partial g(x,y)}{\partial x} - \lambda_1 \cdot q_m \cdot p x^{\ell-1}(1-x)^{n-\ell-1}(\ell - nx) \\
&+\lambda_2 \cdot q_m \cdot p \cdot x^{\ell-2}(1-x)^{n-\ell}(\ell-1-nx) - \lambda_x + \nu_x = 0 \\
&\frac{\partial g(x,y)}{\partial y} + \lambda_1 \cdot (1-q_m) \cdot (1-p) y^{\ell-1}(1-y)^{n-\ell-1}(\ell - ny) \\
&-\lambda_2 \cdot (1-q_m) \cdot (1-p) \cdot y^{\ell-2}(1-y)^{n-\ell}(\ell-1-nx) - \lambda_y + \nu_y = 0 \\
&\lambda_1 \cdot (q_m \cdot p \cdot x^{\ell}(1-x)^{n-\ell} - (1-q_m)(1-p) y^{\ell}(1-y)^{n-\ell} - 0) = 0 \\
&\lambda_2 \cdot (-q_m \cdot p \cdot x^{\ell-1}(1-x)^{n-\ell+1} + (1-q_m)(1-p) y^{\ell-1}(1-y)^{n-\ell+1} - 0) = 0 \\
&\lambda_x \cdot (x-1) = 0 \\
&\lambda_y \cdot (y-1) = 0 \\
&\nu_x \cdot x = 0 \\
&\nu_y \cdot y = 0 \\
&\lambda_1, \lambda_2, \lambda_x, \lambda_y, \nu_x, \nu_y \geq 0
\end{aligned}$$

solving the equations and inequalities yields $x^*_{MD} = \sqrt[n]{\frac{(1-q_m)(1-p)}{q_m p}}, y^*_{MD} = 1$ and $\hat{\ell} = n$. To see the latter, notice that $\forall \ell < n$, both $\gamma(\ell-1, n)$ and $\gamma(\ell, n) \to 0$ as $y \to 1$. Only when $\hat{\ell} = n$ we have $\gamma(\hat{\ell}, n) = (1-p)/(p \cdot x^n + 1 - p) \geq q_m$ while $\gamma(\hat{\ell}-1, n) < q_m$.

Plug $x^*_{MD} = \sqrt[n]{\frac{(1-q_m)(1-p)}{q_m p}}, y^*_{MD} = 1$ into the sender's objective function, we have:

$$\begin{aligned}
\text{Prob}_{\text{MD}}(v = B) &= p \cdot (x_{\text{MD}})^n + (1-p) \cdot (y_{\text{MD}})^n \\
&= p \cdot \frac{(1-q_m) \cdot (1-p)}{q_m \cdot p} + (1-p) = \text{Prob}_{\text{PUB}}(v = B)
\end{aligned}$$

8.3.4 Proof of Proposition 5

Let $x = \pi(b|\alpha)$ and $y = \pi(b|\beta)$ denote the sender's choice of the conditional probabilities. We first show that for all $p \in [0,1]$, public persuasion helps the sender to achieve $\Lambda(p)$ as illustrated in Figure 4. For any given (x, y), let $\underline{p} = 1/(\frac{1-x}{1-y} \cdot \frac{q_m}{1-q_m} + 1)$. We have the following three cases:

Case 1: $p \in [0, \underline{p})$. Since $p < \underline{p} < 1 - q_m$, we have $q_m \leq \mu(b|\alpha) < \mu(b|\beta)$. This means at least m receivers vote for B regardless of the signal realization. Thus the sender gets voting outcome B with probability 1.

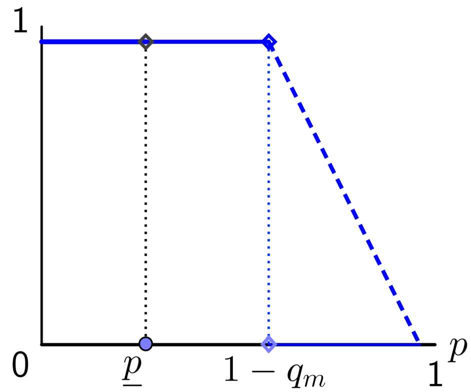

Figure 4: Attainability of the $\Lambda(p)$

Case 2: $p \in [\underline{p}, 1-q_m)$. We have $\mu(b|\alpha) < q_m, \mu(b|\beta) > q_m$, which means more than m receivers vote for B on a signal realization b and less than m receivers vote for B on signal realization a. This yields the sender expected payoffs $(1-p)/q_m$, which is less than $\Lambda(p), \forall p \in [\underline{p}, 1-q_m)$. However, the sender can increase her payoffs by choosing $\pi(b|\alpha) = \pi(b|\beta)$. In this way the signal realizations become completely uninformative and the receivers make decision on the common prior. So the sender get the payoff as high as her expected payoff without persuasion, when the receivers cast votes based on their common prior.

Case 3: $p \in [1-q_m, 1]$. Without persuasion, the sender's expected payoff is always 0. Under public persuasion, however, the sender chooses $\{\pi(\cdot|t)\}_{t\in T}$ as shown in Proposition 1. The sender gets

$$\text{Prob}_{\text{PUB}}(v = B) = (1-p) \cdot \frac{1}{q_m}$$

which is the value of $\Lambda(p), \forall p \in [1-q_m, 1]$.

From Proposition 4 we know that $\text{Prob}_{\text{MD}}(v = B) = (1-p) \cdot \frac{1}{q_m}$. Now it remains to show that the same results holds if the sender is required to generate n correlated, not independent, public signal realizations. Denote the conditional joint distributions of the n signal realizations by η_j, ξ_j, where η_j represents the probability that the sender's persuasion mechanism generates j signal $s = b$ in state α; ξ_j represents the probability that the mechanism generates j signal $s = b$ in state β. Similar to the proof of Proposition

4, there exists cut off number of b signals ℓ such that

$$\gamma(\ell-1,n) = \frac{(1-p)\sum_{j=\ell-1}^{n}\binom{n}{j}\xi_j}{p\sum_{j=\ell-1}^{n}\binom{n}{j}\eta_j + (1-p)\sum_{j=\ell-1}^{n}\binom{n}{j}\xi_j} < q_m$$

$$\gamma(\ell,n) = \frac{(1-p)\sum_{j=\ell}^{n}\binom{n}{j}\xi_j}{p\sum_{j=\ell}^{n}\binom{n}{j}\eta_j + (1-p)\sum_{j=\ell}^{n}\binom{n}{j}\xi_j} \geq q_m$$

and the sender's objective function is

$$p\sum_{j=\ell}^{n}\binom{n}{j}\eta_j + (1-p)\sum_{j=\ell}^{n}\binom{n}{j}\xi_j$$

Solving the maximization and we get $\text{Prob}_{\text{CRR}}(v=B) = (1-p)\cdot\frac{1}{q_m}$, the same expected payoff as $\text{Prob}_{\text{PUB}}(v=B)$.

8.3.5 Proof of Proposition 7

Without loss of generality, we normalize receivers' utilities over correct decisions to 0: $u_i(A,\alpha)=0, u_i(B,\beta)=0$, and utility losses over mistaken decisions to $u_i(A,\beta)=-(1-q_i)$ and $u_i(B,\alpha)=-q_i, \forall i=1,...,n$. The utilitarian receivers' welfare function reduces to

$$W = \frac{1}{n}\cdot\sum_{i=1}^{n}(-q_i)\cdot\ \text{Prob}(v=B|\alpha) + \frac{1}{n}\cdot\sum_{i=1}^{n}-(1-q_i)\cdot\ \text{Prob}(v=A|\beta)$$

First we show that $W_{\text{PRI}} \geq W_{\text{PUB}}$ when the m-th receiver's threshold doubt q_m is sufficiently small. Notice that under public persuasion, the probability of decision errors equals the probability that the mechanism generates a signal $s=b$ in state α plus the probability that the mechanism fails to generate a signal $s=b$ in state β. Then $x^*_{\text{PUB}} = \frac{(1-q_m)\cdot(1-p)}{q_m\cdot p}, y^*_{\text{PUB}}=1$ implies:

$$\begin{aligned} W_{\text{PUB}} &= \frac{1}{n}\cdot\sum_{i=1}^{n}(-q_i)\cdot p\cdot x^*_{\text{PUB}} + \frac{1}{n}\cdot\sum_{i=1}^{n}(-(1-q_i))\cdot(1-p)\cdot(1-y^*_{\text{PUB}}) \\ &= \frac{1}{n}\cdot\sum_{i=1}^{n}(-q_i)\cdot\frac{(1-q_m)\cdot(1-p)}{q_m} \end{aligned}$$

Under private persuasion, due to multiplicity, the receivers' error probability depends on the induced voting profile in equilibrium. If the symmetric voting profile of *Class 3* is induced, the m receivers with smaller threshold doubts vote informatively upon observing

$s = b$, and the rest $n-m$ receivers vote for A uninformatively. Thus there are at most m votes for B. In state α receivers mistakenly implement B only when all the m informative voters observe $s = b$. In state β receivers mistakenly implement A when at least 1 out of all the m receivers' observation is *not* $s = b$. Inserting $x^* = \sqrt[m]{\frac{(1-p)\cdot(1-q_m)}{p\cdot q_m}}, y^* = 1$ we have

$$\begin{aligned} W'_{\text{PRI}} &= \frac{1}{n}\cdot\sum_{i=1}^{n}(-q_i)\cdot p\cdot(x^*)^m + \frac{1}{n}\cdot\sum_{i=1}^{n}(-(1-q_i))\cdot(1-p)\cdot\sum_{j=0}^{m-1}(y^*)^j\cdot(1-y^*)^{m-j} \\ &= \frac{1}{n}\cdot\sum_{i=1}^{n}(-q_i)\cdot\frac{(1-q_m)\cdot(1-p)}{q_m} \end{aligned}$$

In this case, we have $W'_{\text{PRI}} = W_{\text{PUB}}$. Alternatively, if the voting profiles of *Class 2* is induced, there are more than $n-m+1$ receivers vote for A uninformatively. Option B will not be implemented, which is mistaken if the state is β. In this situation we have

$$\begin{aligned} W''_{\text{PRI}} &= \frac{1}{n}\cdot\sum_{i=1}^{n}(-q_i)\cdot p\cdot 0 + \frac{1}{n}\cdot\sum_{i=1}^{n}(-(1-q_i))\cdot(1-p)\cdot 1 \\ &= \frac{1}{n}\cdot\sum_{i=1}^{n}(-(1-q_i))\cdot(1-p) \end{aligned}$$

Therefore, private persuasion yields the receivers' higher welfare if and only if

$$\begin{aligned} & W''_{\text{PRI}} \geq W_{\text{PUB}} \\ \Leftrightarrow\ & \frac{1}{n}\cdot\sum_{i=1}^{n}(-(1-q_i))\cdot(1-p) \geq \frac{1}{n}\cdot\sum_{i=1}^{n}(-q_i)\cdot\frac{(1-q_m)\cdot(1-p)}{q_m} \\ \Leftrightarrow\ & 1 \leq \frac{1}{n}\cdot\sum_{i=1}^{n} q_i\cdot\frac{1}{q_m} \end{aligned}$$

which implies $q_m \leq \sum_{i=1}^{n} q_i/n$. Summarize the two classes of equilibrium outcomes, $W_{\text{PRI}} \geq W_{\text{PUB}}$ if and only if $q_m \leq \sum_{i=1}^{n} q_i/n$.

Next we show that $W_{\text{MD}} = W_{\text{PUB}}$. Plugging $x^*_{\text{MD}}, y^*_{\text{MD}}$ in proposition 4, we get the probability of the receivers' making decision mistakes is:

$$\begin{aligned} W_{\text{MD}} &= \frac{1}{n}\cdot\sum_{i=1}^{n}(-q_i)\cdot p\cdot(x_{\text{MD}})^n + \frac{1}{n}\cdot\sum_{i=1}^{n}(-(1-q_i))\cdot(1-p)\cdot\sum_{j=0}^{\ell-1}(y_{\text{MD}})^j(1-y_{\text{MD}})^{n-j} \\ &= \frac{1}{n}\cdot\sum_{i=1}^{n}(-q_i)\cdot\frac{(1-q_m)\cdot(1-p)}{q_m} = W_{\text{PUB}} \end{aligned}$$

8.3.6 Proof of Lemma 3

We follow Duggan and Martinelli's Duggan and Martinelli [2001] approach in this proof. Suppose the voting strategy $\sigma_i(s)$ described in Lemma 3 is not a best response for receiver i. Then there is another σ_i' that yields receiver i expected payoff as least as high as the expected payoff from σ_i. Specifically, consider two sets

$$V = \{s \in [0,1] | \Delta(\sigma_{-i}, s) < 0 \text{ and } \sigma_i'(s) < 1\}$$
$$W = \{s \in [0,1] | \Delta(\sigma_{-i}, s) > 0 \text{ and } \sigma_i'(s) > 0\}$$

The difference of expected payoffs between strategy σ_i' and σ_i is

$$\int_V (\sigma_i' - 1)(p(-q_i)P_{-i}(\text{piv}|\alpha)\pi(s|\alpha) - (1-p)(-(1-q_i))P_{-i}(\text{piv}|\beta)\pi(s|\beta))ds$$
$$+ \int_W \sigma_i'(p(-q_i)P_{-i}(\text{piv}|\alpha)\pi(s|\alpha) - (1-p)(-(1-q_i))P_{-i}(\text{piv}|\beta)\pi(s|\beta))ds$$

When $s \in V, \Delta(\sigma_{-i}, s) < 0$ and $\sigma_i' - 1 < 0$, so the first term of the integral is negative. When $s \in W, \Delta(\sigma_{-i}, s) > 0$ and $\sigma_i' > 0$, so the second term is also negative. Thus we get the contradiction, which indicates that the above specified voting strategy σ_i is a best response given σ_{-i}.

$\Delta(\sigma_{-i}, s)$ is continuous in s. Moreover, Assumption 2 implies that it is strictly decreasing in s. Thus it follows that there is a unique cutoff signal $\bar{s}_i$ for receiver i which solves $\Delta(\sigma_{-i}, s) = 0$ and $\inf\{s \in [0,1] | \Delta(\sigma_{-i}, s) \leq 0\}$ is a well defined. Therefore, σ_i being the best response implies that it is equivalent to the cutoff strategy $\hat{\sigma}_i$.

8.3.7 Proof of Lemma 4

Since $\Delta(\sigma_{-i}, s)$ is continuous in q_i, so does $\bar{s}_i$. Take any two threshold doubts $0 \leq q_j \leq q_i \leq 1$. We have

$$\Delta(\sigma_{-i}, s) = P_{-i}(\text{piv}|\alpha) \cdot \pi(s|\alpha) \cdot q_i \cdot p - P_{-i}(\text{piv}|\beta) \cdot \pi(s|\beta) \cdot (1-q_i) \cdot (1-p)$$
$$\Delta(\sigma_{-j}, s) = P_{-j}(\text{piv}|\alpha) \cdot \pi(s|\alpha) \cdot q_j \cdot p - P_{-j}(\text{piv}|\beta) \cdot \pi(s|\beta) \cdot (1-q_j) \cdot (1-p)$$

and the following cutoff strategies for i and j, respectively:

$$\sigma_i(s) = \begin{cases} 1, & \text{if } s \geq \bar{s_i} \\ 0, & \text{if } s < \bar{s_i} \end{cases}$$

where $\bar{s_i} = \inf\{s \in [0,1] | \Delta(\sigma_{-i}, s) \leq 0\}$.

$$\sigma_j(s) = \begin{cases} 1, & \text{if } s \geq \bar{s_j} \\ 0, & \text{if } s < \bar{s_j} \end{cases}$$

where $\bar{s_j} = \inf\{s \in [0,1] | \Delta(\sigma_{-j}, s) \leq 0\}$.

Suppose $\bar{s_j} = \bar{s_i}$. Since $\Delta(\sigma_{-j}, s)$ is continuous in s and q, there exists $\epsilon > 0$ such that at $s = \bar{s_i} - \epsilon, \Delta(\sigma_{-i}, s) > 0$, so receiver i will vote for B with probability 0. However, $s = \bar{s_i} - \epsilon = \bar{s_j} - \epsilon$ and $q_j < q_i$ implies $\Delta(\sigma_{-j}, s) \leq 0$, which indicates that receiver j is not willing to vote for A. The cutoff strategy $\sigma_j(s)$ is not a best response. A contradiction.

Now suppose $\bar{s_j} > \bar{s_i}$. Take $s \in [\bar{s_i}, \bar{s_j})$. Since $s \geq \bar{s_i}$, we have $\Delta(\sigma_{-i}, s) \leq 0$, so receiver i is willing to vote for B with probability 1. Moreover, $s < \bar{s_j}$ implies $\Delta(\sigma_{-j}, s) > 0$, which means receiver j will vote for A. Nonetheless, $q_j < q_i$ implies $\Delta(\sigma_{-j}, s) < \Delta(\sigma_{-i}, s) \leq 0$, which indicates receiver j is not willing to vote for A. A contradiction. Thus the first part of the lemma follows.

For the second part of the lemma, notice that for each receiver i, $\bar{s_i} = \inf\{s \in [0,1] | \Delta(\sigma_{-i}, s) \leq 0\}$. Expand the inequality $\Delta(\sigma_{-i}, s) \leq 0$, we have:

$$\begin{aligned} & \sum_{|M|=m-1, i \notin M} (\prod_{j \in M} (\int_0^1 \sigma_i(s) \frac{\pi(s|\alpha)p(\alpha)}{\sum_{t' \in T} \pi(s|t')p(t')} ds) \prod_{j \notin M, j \neq i} (\int_0^1 (1-\sigma_i(s)) \frac{\pi(s|\alpha)p(\alpha)}{\sum_{t' \in T} \pi(s|t')p(t')} ds)) \pi(s|\alpha) \cdot q_i \cdot p \\ \leq & \sum_{|M|=m-1, i \notin M} (\prod_{j \in M} (\int_0^1 \sigma_i(s) \frac{\pi(s|\beta)p(\beta)}{\sum_{t' \in T} \pi(s|t')p(t')} ds) \prod_{j \notin M, j \neq i} (\int_0^1 (1-\sigma_i(s)) \frac{\pi(s|\beta)p(\beta)}{\sum_{t' \in T} \pi(s|t')p(t')} ds)) \pi(s|\beta) \cdot (1-q_i) \cdot (1-p) \end{aligned}$$

Denote the ratio of the probability densities as $r = \frac{\pi(s|\alpha)}{\pi(s|\beta)}$. As r increases, the LHS of the inequality increases while the RHS decreases or remain unchanged. In both cases, for some $s \in [\bar{s_i}, 1]$, $\Delta(\sigma_{-i}, s) > 0$. Continuity implies that such $s's$ either satisfy $s \in [\bar{s_i}, \bar{s_i} + \eta]$ or satisfy $s \in [1 - \vartheta, 1]$. Suppose for $s \in [1 - \vartheta, 1]$ we have $\Delta(\sigma_{-i}, s) > 0$. But this contradicts the definition of a *cutoff strategy*. Thus for $s \in [\bar{s_i}, \bar{s_i} + \eta]$ the value of $\Delta(\sigma_{-i}, s)$ are positive. Hence the value of $\inf\{s \in [0,1] | \Delta(\sigma_{-i}, s) \leq 0\}$ increases.

8.3.8 Proof of Corollary 8

Under public persuasion, the sender chooses a set of conditional distributions $(\pi(s|\alpha), \pi(s|\beta))$ over $S = [0, 1]$ to maximize the probability that $s \geq \bar{s_m}$:

$$\max_{(\pi(s|\alpha),\pi(s|\beta))} p \cdot P(s \geq \bar{s_m}|\alpha) + (1-p) \cdot P(s \geq \bar{s_m}|\beta)$$

The receivers' posterior belief after observing a signal realization $s \geq \bar{s_m}$ is:

$$\begin{aligned} \mu(\beta|s \geq \bar{s_m}) &= \frac{(1-p) \cdot P(s \geq \bar{s_m}|\beta)}{p \cdot P(s \geq \bar{s_m}|\alpha) + (1-p) \cdot P(s \geq \bar{s_m}|\beta)} \\ &= \frac{(1-p) \cdot \int_{\bar{s_m}}^{1} \pi(s|\beta)ds}{p \cdot \int_{\bar{s_m}}^{1} \pi(s|\alpha)ds + (1-p) \cdot \int_{\bar{s_m}}^{1} \pi(s|\beta)ds} \end{aligned}$$

Having at least m receivers voting for B upon observing $s \geq \bar{s_m}$ requires $\mu(\beta|s \geq \bar{s_m}) \geq q_m$. Thus the sender's expected utility is maximized when the conditional distributions $(\pi(s|\alpha), \pi(s|\beta))$ satisfy:

$$(1-p) \cdot (1-q_m) \cdot \int_{\bar{s_m}}^{1} \pi(s|\beta)ds \geq p \cdot q_m \cdot \int_{\bar{s_m}}^{1} \pi(s|\alpha)ds$$

and $EU^S = p \cdot \frac{(1-p)(1-q_m)}{p \cdot q_m} + (1-p) = \Lambda(p), \forall p \in [0, 1]$.

References

D. Austen-Smith and J.S. Banks. Information aggregation, rationality, and the condorcet jury theorem. *American Political Science Review*, pages 34–45, 1996.

D. Austen-Smith and T.J. Feddersen. Information aggregation and communication in committees. *Philosophical Transactions of the Royal Society B: Biological Sciences*, 364 (1518):763–769, 2009.

David Austen-Smith and T. Feddersen. Deliberation and voting rules. *The Center for Mathematical Studies in Economics and Management Science at Northwestern University Discussion Paper No. 1359*, 2002a.

David Austen-Smith and T. Feddersen. The inferiority of deiliberaition under unanimity. *The Center for Mathematical Studies in Economics and Management Science at Northwestern University Discussion Paper No. 1359*, 2002b.

B. Caillaud and J. Tirole. Consensus building: How to persuade a group. *The American Economic Review*, 97(5):1877–1900, 2007.

A. Chakraborty and R. Harbaugh. Persuasion by cheap talk. *The American Economic Review*, 100(5):2361–2382, 2010.

P.J. Coughlan. In defense of unanimous jury verdicts: Mistrials, communication, and strategic voting. *American Political Science Review*, pages 375–393, 2000.

V.P. Crawford and J. Sobel. Strategic information transmission. *Econometrica: Journal of the Econometric Society*, pages 1431–1451, 1982.

S. DellaVigna and M. Gentzkow. Persuasion: Empirical evidence. *National Bureau of Economic Research*, 2009.

J. Duggan and C. Martinelli. A bayesian model of voting in juries. *Games and Economic Behavior*, 37(2):259–294, 2001.

J. Farrell and R. Gibbons. Cheap talk with two audiences. *The American Economic Review*, 79(5):1214–1223, 1989.

T. Feddersen and W. Pesendorfer. Voting behavior and information aggregation in elections with private information. *Econometrica: Journal of the Econometric Society*, pages 1029–1058, 1997.

T. Feddersen and W. Pesendorfer. Convicting the innocent: The inferiority of unanimous jury verdicts under strategic voting. *American Political Science Review*, pages 23–35, 1998.

T. Feddersen and W. Pesendorfer. Elections, information aggregation, and strategic voting. *Proceedings of the National Academy of Sciences*, 96(19):10572, 1999.

D. Gerardi and L. Yariv. Deliberative voting. *Journal of Economic Theory*, 134(1):317–338, 2007.

J. Glazer and A. Rubinstein. On optimal rules of persuasion. *Econometrica*, 72(6):1715–1736, 2004.

J. Glazer and A. Rubinstein. A study in the pragmatics of persuasion: a game theoretical approach. *Theoretical Economics*, 1(4):395–410, 2006.

M. Goltsman and G. Pavlov. How to talk to multiple audiences. *Games and Economic Behavior*, 72(1):100–122, 2011.

J.R. Green and J.J. Laffont. Partially verifiable information and mechanism design. *The Review of Economic Studies*, pages 447–456, 1986.

J.R. Green and N.L. Stokey. A two-person game of information transmission. *Journal of Economic Theory*, 135(1):90–104, 2007.

S.J. Grossman. The informational role of warranties and private disclosure about product quality. *Journal of Law and economics*, 24(3):461–483, 1981.

E. Kamenica and M. Gentzkow. Bayesian persuasion. *The American Economic Review*, 101(6):2590–2615, 2011a.

Emir Kamenica and Matthew Gentzkow. Competition in persuasion. *University of Chicago Working Paper*, 2011b.

K.K. Ladha. The condorcet jury theorem, free speech, and correlated votes. *American Journal of Political Science*, pages 617–634, 1992.

R.P. McAfee and J. McMillan. Multidimensional incentive compatibility and mechanism design. *Journal of Economic Theory*, 46(2):335–354, 1988.

A. McLennan. Consequences of the condorcet jury theorem for beneficial information aggregation by rational agents. *American Political Science Review*, pages 413–418, 1998.

A. Meirowitz. In defense of exclusionary deliberation: communication and voting with private beliefs and values. *Journal of Theoretical Politics*, 19(3):301–327, 2007.

P. Milgrom and J. Roberts. Relying on the information of interested parties. *The RAND Journal of Economics*, pages 18–32, 1986.

P.R. Milgrom. Good news and bad news: Representation theorems and applications. *The Bell Journal of Economics*, pages 380–391, 1981.

Luis Rayo and Ilya Segal. Optimal information disclosure. *Stanford University, Department of Economics Working Paper*, 2010.

Frank Rosar and Elisabeth Schulte. Imperfect private information and the design of information-generating mechanisms. *University of Bonn Working Paper*, 2010.

E. Schulte. Communication in committees: who should listen? *Public Choice*, pages 1–21, 2011.

M. Spence. Job market signaling. *The quarterly journal of Economics*, 87(3):355–374, 1973.

Ina Taneva. Informational design. *Working Paper*, 2014.