A ROBUST EQUILIBRIUM RELATIONSHIP BETWEEN MARKET PRICES OF RISKS AND RISK AVERSION IN DYNAMICALLY COMPLETE STOCHASTIC VOLATILITY MODELS*

QIAN HAN[1]†, CALUM G. TURVEY[2]

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Abstract

We derive a general equilibrium linear relationship between the market prices of risks and market risk aversion under a continuous time stochastic volatility model completed by liquidly traded options. The relation is robust as it is valid for both endowment and production economies, and for both regular time-separable von-Neumann Morgenstern and non-time-separable habit formation preferences. The relation can be used in practice to construct a daily market risk aversion index from options market.

JEL CLASSIFICATION: C61, D51, G11, G13

KEYWORDS: General equilibrium, market price of risk, market risk aversion, market pricing kernel, habit formation, stochastic volatility model

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[1] The Wang Yanan Institute for Studies in Economics (WISE), MOE Key Laboratory of Econometrics, and Fujian Key Laboratory of Statistical Sciences, Xiamen University.

[2] Charles H. Dyson School of Applied Economics and Management, Cornell University..

†:Corresponding author, contact address: Room A402, Economics Building, The Wang Yanan Institute for Studies in Economics (WISE), Xiamen University, Xiamen, Fujian Province, China, 361005. Tel: +86 592 218 0969, Fax: +86 592 218 7708, E-mail: wise.hanqian@gmail.com.

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Abstract

We derive a general equilibrium linear relationship between the market prices of risks and market risk aversion under a continuous time stochastic volatility model completed by liquidly traded options. The relation is robust as it is valid for both endowment and production economies, and for both regular time-separable von-Neumann Morgenstern and non-time-separable habit formation preferences. The relation can be used in practice to construct a daily market risk aversion index from options market.

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1. Introduction

It is well known that in a continuous time general equilibrium model with complete market the factor risk premium is linked to market risk aversion (MRA) through the covariance between the risk factor and changes in aggregate consumption (Breeden (1979), Cox, Ingersoll and Ross (1985), Duffie (1992) among others). Given the notorious difficulty in measuring aggregate consumption, however, the relation is more of academic interest than any practical use. In this paper, we re-examine the relation by considering a security market that includes (1) a *real* security whose price evolution is described by a stochastic volatility model and (2) *financial* securities, namely a riskless bond and options written on the real security[3]. We find that in equilibrium the instantaneous MRA is a linear combination of the two market prices of risks – one is the market price of asset risk (MPAR), the other the market price of orthogonal risk (MPOR) [4]. The result is consistent with the intuition that the market prices of risks, as a bridge connecting the risk neutral and the subjective world, should contain information about market investors' risk preferences.

While we establish the model we try to make it as realistic as possible. For that purpose we consider a production economy, as in comparison to an endowment economy, to accommodate the real world observation that dividends are usually exogenously determined. In addition, the number of stock shares is in reality not a constant but time-varying as companies issue new shares to or repurchase old shares from the market. We also consider a class of non-time-separable habit formation preferences which is arguably more realistic than a usual time-separable von Neumann Morgenstern type. Many studies (Hansen and Singleton (1982, 1983), Ferson (1983), Grossman et al. (1987), Ferson and Constantinides (1989), Hansen and Jagannathan (1988), Hall (1988), Heaton (1988)) reject the time-separable models and support the existence of habit persistence. In addition, habit formation has been successful in resolving the well documented equity premium puzzle (Constantinides (1990), Campbell and Cochrane (1999)) by relaxing the constraint that the product of the elasticity of substitution and risk aversion has to be unity under a time-separable utility. By assuming a habit formation preference for market investors, we allow for the path dependency of the market pricing kernel and risk aversion.

Our combination of a production economy with a habit formation preference avoids the problem as demonstrated in Chapman (1998). He shows that when one adopts an endowment economy to examine equilibrium asset prices under habit formation, the

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[3] Recent studies (Anderson and Raimondo (2008), Hugonnier, Malamud and Trubowitz (2009)) have shown that such a market admits equilibrium with dynamic completeness under rather mild restrictions on the model primitives. We leave more detailed discussions on endogenous completeness to another section.

[4] It is "orthogonal" because it is the risk premium associated with the random source of volatility which is orthogonal to the randomness driving the underlying asset price process. While not obvious, the MPOR is consistent with the traditional definition of market price of risk as excess return per unit of risk of a zero-beta hedging portfolio. See Lewis (2000) for a demonstration.

model parameters, when calibrated to match the first two moments of the aggregated consumption data, can possibly give rise to a negative market pricing kernel. One solution, as he suggests, is to adopt a production economy with sufficient structure on the asset return distribution.

We start by considering the equilibrium relation between market prices of risks and the market pricing kernel. Specifically, in a stochastic volatility model, the market price of asset (orthogonal) risk is found in equilibrium to be a linear combination of the partial derivatives of the market pricing kernel with respect to the underlying asset and the stochastic variance. This makes intuitive sense because a pricing kernel is a continuous version of Arrow-Debreu security prices, the derivative of which with respect to a risk factor reflects the market compensation in unit of utility for taking additional risk in that factor. It is interesting to find that, if we plug our derived equilibrium relation between prices of risks and pricing kernel to the popular partial differential equation for contingent claim prices we can recover the fundamental valuation formula in Garman (1976).

Our expression of the relation between prices of risks and the pricing kernel is similar to that derived in He and Leland (1993) and Pham and Touzi (1996) although the focus of those studies is not the relation per se but the consistency of the state price system with the economic equilibrium, or equivalently the viability of risk premiums. He and Leland (1993) consider a pure exchange economy endowed with one unit of risky asset. Dividends are ignored and the risk free rate is treated exogenous as there is no intermediate consumption[5]. They find a partial differential equation condition for the risk premium as well as the equilibrium relation between risk premium and the pricing kernel on the terminal date. Examples of constant volatility are given to illustrate how specifications of utility function affect the equilibrium form of excess return. Pham and Touzi (1996) consider a similar economy but introduce stochastic volatility. The market is completed by a contingent claim on the underlying under some regularity conditions. They find necessary and sufficient conditions for viable risk premium functions and relate them to the pricing kernel in the case of positive dividends[6]. In contrast to He and Leland (1993), they illustrate with examples how assumptions on risk premium can be supported by interim and terminal utility functions. Lewis (2000) instead considers a production economy within the framework of Cox et al. (1985) and also stochastic variance. Assuming power utility throughout, he derives a partial differential equation for risk premium coefficient. Examples are given to illustrate viable forms of risk premium corresponding to different assumptions on the variance process. It is worthy to note that all these studies, including ours, find exactly the same relation between prices of risks and pricing kernel in equilibrium despite the different underlying economic settings considered in each paper.

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5 He and Leland (1993) discuss the case involving intermediate consumption and dividends as an extension.

6 Their results cannot be derived as a case of two risky assets in He and Leland because the total supply of the underlying risky asset is constrained to be one unit in Pham and Touzi [1996].

We then proceed and define MRA as the Arrow-Pratt relative risk aversion defined in terms of the indirect utility function. We solve the representative agent's optimization problem and derive a relation between MRA and the pricing kernel[7]. Using pricing kernel as a bridge, we finally derive a linear relation between the market prices of risks and MRA. Bollerslev, Tauchen and Zhou (2010) give a similar relation between market price of risk and risk aversion. However, their result is based on ad hoc economic arguments with assumptions made on the variance process and a power utility form for the representative agent. In fact, it is shown in Section 2 that with these two assumptions the equilibrium MPOR equals zero, i.e. the orthogonal risk is not priced by the market[8].

Our result is surprisingly robust. While we enrich the economic structure from an endowment economy with usual time-separable utility functions to a production economy with habit formation, we get the same equilibrium relation. The result is also simple without involving aggregate consumption data. The two major variables in the equation are the two market prices of risks, which can be extracted from market option prices by model calibration. Hence we provide a new and practical approach to extract MRA from option prices.

Many previous studies to extract MRA rely on comparing the estimations of risk neutral density and subjective probability density of the underlying asset on option maturity date T (for example, implied binomial tree approach in Jackwerth and Rubinstein (1996), Jackwerth (2000), and non-parametric approach in Ait-Sahalia and Lo (2000)). However, these estimates either fail to incorporate stochastic volatility or assume a too simple economic setting. Most importantly, they are all estimates of the *intertemporal* risk aversion between the current and the option maturity date. The relation we derive in this paper allows a direct computation of the *instantaneous* MRA as a linear function of the implied MPAR and MPOR. Also note that a habit formation preference allows a time varying MRA by introducing a habit level.

The rest of the paper is organized as follows. Section 2 describes the production economy, model assumptions and derives the equilibrium relations between market prices of risks, the market pricing kernel and risk aversion using a dynamic programming approach. Section 3 discusses the implications of these results and provides several examples. Empirical study on the cross section and time series of market risk aversion is conducted in Section 4. Section 5 concludes.

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7 A common approach adopted in the literature to relate risk aversion to the pricing kernel is to write $G(t)=\frac{U_c(c,t)}{U_c(c,0)}$, then define the risk aversion $\gamma(t)=-c(t)\frac{U_{cc}(c,t)}{U_c(c,t)}$, hence $\gamma(t)=-s(t)\frac{G_s(t)}{G(t)}$. However, the derivation requires assuming an exchange economy so that in equilibrium the underlying stock equals the consumption, which is not true in a production economy.

8 A zero price of orthogonal risk corresponds to the minimal equivalent martingale $\hat{\mathrm{P}}$-measure proposed by FÖllmer and Schweizer (1991). Pham and Touzi (1996) show that equilibrium supporting utility functions corresponding to a $\hat{\mathrm{P}}$-measure can only take log utility form. The "discrepancy" is due to the different assumptions on dividend policy in that Pham and Touzi (1996) consider a pure exchange economy while the papers mentioned above follow Cox et al. (1985) production economy.

2. Model

2.1 Model Setup

In this section we follow Cox et al. (1985) and consider a production economy[9] where capital goods are invested to solely produce a risky asset S_t with a linear technology. The production process is captured by a stochastic differential equation with the drift and volatility terms dependent on state variables. There are potentially many state variables that can affect the production process. However, for simplification we assume that there is only one relevant state variable, the variance of the risky asset ς_t which follows another stochastic differential equation specified in

Assumption 1: *The joint production and state variable process* (S_t, ς_t) *form a stochastic volatility model (SVM):*

$$
\begin{aligned}
dS_t &= [\mu_1(t,S_t,\varsigma_t) - D(t,S_t,\varsigma_t)]dt + \sigma_1(t,S_t,\varsigma_t)dW_{1t} \\
d\varsigma_t &= \mu_2(t,S_t,\varsigma_t)dt + \sigma_2(t,S_t,\varsigma_t)dZ_t \qquad (2.0) \\
dZ_t &= \rho_t dW_{1t} + \sqrt{1-\rho_t^2}\, dW_{2t}
\end{aligned}
$$

where $\varsigma_t = \sigma_t^2$, ρ_t is the instantaneous correlation coefficient between the two Brownian motions W_{1t} and Z_t. The drift and volatility terms for both the risky asset

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[9] The probabilistic structure of the economy is based on a complete probability space (Ω, F, P) where Ω is the event space with a typical element ω, F the sigma algebra of observable events, P a probability measure assigned on (Ω, F). Let I_t denote the information set faced by investors at time t, then the payoff space where π_t is the asset payoff at time t. That is, the payoff space is the set of all random variables with finite conditional second moments given the previous period information. For the model we consider a market consisting of risky assets (the underlying stock and European options written on it) and a riskless asset with an instantaneous interest rate r. We use Γ_t to denote the payoff space on the market at time t, so $\Gamma_t \subset \Phi$ for all time t.

and its variance are all assumed to be real analytic functions of the state space. The dividend process $D_t = \delta_t S_t$ with dividend rate δ_t. As in Lewis (2000) we treat the dividend policy as an exogenous variable to allow for the variance of stock shares. This also facilitates the empirical study in the next section where dividends are taken directly from market observations. Note that we leave the variance process unspecified so square root, 3/2 models are included.

Before we analyze investors' portfolio choice problem over a finite horizon [0, T], we need to specify the investment opportunity set.

Assumption 2: *Investors can borrow and lend a riskless asset B at an interest rate r, which is determined endogenously in equilibrium.*

Besides the riskless asset B, there are also traded European options $\{V_{ij}\}$ with time to maturities $0 \le T_1 \le ... T_i \le ... \le T_M = T$. Let N_{T_i} denote the number of strikes for the contract with time to maturity T_i so there are $\sum_{i=1}^{M} N_{T_i}$ traded European options.

Assumption 3: *There exists at least one traded European put or call options F written on the underlying asset such that a set composed of the riskless asset B, the risky underlying asset S and the option form a basis*[10].

Assumption 3 is equivalent to assume that the exogenous model coefficients in (2.0) are such that at least one of these put or call options completes the market in the sense of endogenous completeness as shown in recent works by Anderson and Raimondo (2008) and Hugonnier, Malamud and Trubowitz (2009).

Previous studies on equilibrium asset prices in a continuous time model ensure existence of equilibrium by aggregating individual agents into a representative agent. They either focus on a complete market directly (He and Leland (1993)), or complete the market with traded contingent claims under restrictive conditions and strong assumptions on the model coefficients (Pham and Touzi (1996)[11]), or assume identical

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[10] A basis is defined in Cox et al. (1985) as a set of production processes and a set of contingent claims that span the market.

[11] Since Pham and Touzi (1996), there have been advances in completing the market in continuous time models. Romano and Touzi (1997) show that under restrictive conditions on the drift and volatility coefficients in the state variable process and the volatility risk premium, any European contingent claim completes the market. Davis and Obloj (2007) extend conditions proposed in Davis (2004) to a necessary and sufficient condition for market completeness using vanilla or path-dependent derivatives (Theorem 4.1 in the article where an invertibility condition is imposed on the gradient matrix of the pricing functions with respect to the underlying state variables). In particular, they show that for a

individual preferences (Lewis (2000)). Recently, Anderson and Raimondo (2008) show that, a sufficient condition for the existence of equilibrium with dynamic completeness is the exogenous non-degeneracy of the terminal dividends at some point in the state space. In particular, European options can be included in the basis securities. Hugonnier et al. (2009) further relax the restriction of non-degeneracy on *terminal* dividends and allow for an infinite horizon (their theorem 3). They also permit the inclusion of a fixed income security to the basis securities by considering a second order expansion of the volatility matrix (their theorem 2). Both articles point out that the non-degeneracy condition only depends on the dividends, not other economic primitives such as utility functions or initial endowments. The major conclusion of these studies is that for most continuous time financial models in the literature equilibrium with dynamically complete market exists for very general assumptions on the volatility matrix of dividends. Since the model used in our paper is pretty standard and we treat the asset process as exogenous, we are more confident than previous studies to ensure existence of equilibrium with dynamic completeness. For illustration purposes, in the following we use an arbitrary valid put option to complete the market.

It follows immediately that the market admits no arbitrage and a unique and strictly positive market wide pricing kernel $G(t,T)$ exists on the payoff space Γ_t with $G(t)=\exp\left\{-\int_0^t r_u du-\frac{1}{2}\int_0^t(\lambda^2_{1u}+\lambda^2_{2u})du-\int_0^t\lambda_{1u}dW_{1u}-\int_0^t\lambda_{2u}dW_{2u}\right\}$ where λ_{1u} and λ_{2u} are adapted processes of MPAR and MPOR respectively. Define a risk neutral Q-measure by $\left.\frac{dQ}{dP}\right| I_t = e^{\int_0^t r_s ds}G(t)$ so $p_t = E^Q[e^{-r(T-t)}\Pi_T \,|\mathrm{I}_t]$. Using the fact that the joint stock and volatility process $\{S_t,\varsigma_t\}$ is Markovian, we can write the put option price as $V(t,S_t,\varsigma_t)=E^Q\left[e^{-\int_t^T r_u du}(K-S_T)^+\,|F_t\right]$. Define a new two-dimensional

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stochastic volatility model, if the drift and volatility coefficients in a Markov state variable process are both analytic, and other loose conditions ((4.3) and (A4) in the article) hold, then the market can be completed by any bounded European contingent claim such as a put option (Proposition 5.1 in the article). Using a slightly different approach, Jacod and Protter (2007) show that an unstable complete market may be obtained by imposing some complex compatibility conditions between the underlying asset price and option prices. Carr and Sun (2007) suggest that under certain hypothesis the market can be completed by variance swaps now liquidly traded in the market.

Brownian motion $(\tilde{W}_{1t}, \tilde{W}_{2t})$ as $\tilde{W}_{1t} = W_{1t} + \int_0^t \lambda_{1u} \mathrm{du}, \tilde{W}_{2t} = W_{2t} + \int_0^t \lambda_{2u} \mathrm{du}$. The risk neutral version of the stochastic volatility model of (2.0) is then

(2.1)

$$dS_t = [\mu_{1t} - D_t - \lambda_{1t}\sigma_{1t}]dt + \sigma_{1t} d\tilde{W}_{1t}$$
$$d\varsigma_t = [\mu_{2t} - (\rho_t \lambda_{1t} + \sqrt{1-\rho_t^2}\lambda_{2t})\sigma_{2t}]dt + \rho_t \sigma_{2t} d\tilde{W}_{1t} + \sqrt{1-\rho_t^2}\sigma_{2t} d\tilde{W}_{2t}$$

The term $(\rho_t \lambda_{1t} + \sqrt{1-\rho_t^2}\lambda_{2t})\sigma_{2t}$ is the instantaneous volatility risk premium widely studied in the literature, and $\rho_t \lambda_{1t} + \sqrt{1-\rho_t^2}\lambda_{2t}$ is the instantaneous market price of volatility risk (MPVR). Applying Ito's lemma to the put option price, we have $dV_t = \mu_3(t, V_t)dt + F(t, s, \varsigma)d\hat{W}_t$ where $F(t, s, \varsigma) = \sqrt{V_s^2 \sigma_1^2 + V_\varsigma^2 \sigma_2^2 + 2\rho V_s V_\varsigma \sigma_1 \sigma_2}$ and $\hat{W}_t$ is a Brownian motion under risk-neutral measure with $dW_{1t} d\hat{W}_t = \rho' dt$, $\rho' = \dfrac{\rho V_\varsigma \sigma_2 + V_s \sigma_1}{F}$.

Assumption 4: *The representative agent's preference exhibits habit formation as described by:*

$$E\left[\int_0^T e^{-\int_0^t \varphi_s ds} u(c_t, x_t)dt\right], where\ x_t = x_0 e^{-k_1 t} + k_2 \int_0^t e^{-k_1(t-s)} c_s ds$$ is the habit level.[12]

Habit level x_t is an exponentially weighted average of past consumptions hence making the utility function path dependent. Ito's Lemma shows that the habit level x_t follows a differential equation $dx_t = (k_2 c_t - k_1 x_t)dt, x_0 \geq 0.$ Assuming strict positive marginal utility at time 0 we consider an *additive* habit formation preference, i.e. the agent is forced to consume more than she does in the past. The utility function is increasing and strictly concave in c, strict decreasing in x and concave in (c, x).[13]

[12] Refer to Sundaresan(1989), Constantinides(1990),Detemple and Zapatero(1991), Chapman(1998), Campbell and Cochrane(1999), Detemple and Karatzas(2002) for various versions of the habit level and the utility form.

[13] See Detemple and Zapatero (1991) for a full characterization of the utility function.

Under this structure, an increase in consumption at date t decreases contemporaneous marginal utility but increases the marginal utility of consumption at future dates.

The representative agent's investment opportunity set includes: the stock, the put option and a riskless bond, the latter two being purely financial assets with zero net supply. Let α_t denote the amount invested in the underlying stock, β_t the amount invested in the put option and e_t her total wealth at time t. The agent's consumption is described by the pair of consumption rate process $\{c_t\}$ and final wealth e_T. The wealth process is then expressed through her portfolio process $(\alpha_t, \beta_t, e_t - \alpha_t - \beta_t)$:

$$(2.2) \qquad de_t = \left[\alpha \frac{\mu_1}{S} + \beta \frac{\mu_3}{V} + (e - \alpha - \beta) r - c \right] dt + \alpha \frac{\sigma_1}{S} dW_1 + \beta \frac{F}{V} d\hat{W}$$

A consumption and portfolio strategy (c_t, α_t, β_t) is *feasible* if it satisfies (2.2) with a nonnegative wealth process. We let $\Psi(S_0)$ denote the set of all feasible consumption-portfolio strategies with initial wealth S_0. The representative agent then pursues the following investment problem:

$$(2.3) \qquad \max_{(\alpha_t, \beta_t, c_t) \in \Psi(S_0)} E[\int_0^T u(t, c_t, x_t) dt + U(e_T)]$$

2.2. General Equilibrium

We say that the market is in equilibrium if the representative agent optimally chose to hold only underlying stocks (total shares normalized to be unity) and zero units of options. That is, the market clears for all t: $\alpha_t^* = S_t, \beta_t^* = 0 \ \forall t \in [0,T]$. It follows that in equilibrium the total wealth $e_t^* = S_t \ \forall t \in [0,T]$. Proposition1 states the equilibrium relation between market prices of risks and the pricing kernel:

Proposition 1*: In an economy as specified above, in equilibrium*

$$(1) \ G(t) = \frac{J_e(t)}{J_e(0)} = \frac{u_c\left(t, c^*(t)\right) + k_2 J_x(t)}{u_c\left(0, c^*(0)\right) + k_2 J_x(0)}$$

$$G(T) = \frac{U_S(T, S_T)}{U_s(0, S_0)}$$

where $J(.)$ is the value function of the optimization problem (2.3) and C is the equilibrium consumption that will be solved endogenously. This is equivalent to state that in equilibrium the pricing kernel $G(t)$ can be supported by some utility functions.*

(2) If $(\lambda_{1t}, \lambda_{2t})$ are viable, then:

$$\lambda_1(t) = -\sigma_1(t)G_s(t)/G - \rho(t)\sigma_2(t)G_\varsigma(t)/G$$
$$\lambda_2(t) = -\sqrt{1-\rho(t)^2}\sigma_2(t)G_\varsigma(t)/G$$

with terminal conditions:

$$\lambda_1(T) = -\sigma_1 U_{ss}(T)/U_s(T)$$
$$\lambda_2(T) = 0$$

Proposition 1 can be easily generalized to multifactor pure diffusion models, for example stochastic interest rate can be added. In multifactor cases we have $\vec{\lambda}(t) = -\sigma(t)^T \bullet \nabla' G(t)/G(t)$. ∇G_t is the gradient of G with respect to the risk factors at time t, σ the volatility matrix of the state variable process, the superscript T means transpose of the matrix. Condition (1) states that the pricing kernel in equilibrium can be supported by some utility function, hence can be interpreted as the representative agent's marginal rate of substitution. Condition (2) is similar to those derived in Pham and Touzi (1996) with the only difference being that their expression is in terms of two new processes transformed from MPRs.

Now we derive the partial differential equations that must be satisfied by viable market prices of risks. For brevity we only present results assuming a time-separable utility. Comparing the drift terms in (2.12) and (2.13) in Appendix A we have:

(2.14)

$$G_t + (\mu_1 - D)G_s + \mu_2 G_\varsigma + \frac{1}{2}G_{ss}\sigma_1^2 + G_{s\varsigma}\rho\sigma_1\sigma_2 + \frac{1}{2}G_{\varsigma\varsigma}\sigma_2^2 + rG = 0$$

This gives the equilibrium interest rate in terms of the pricing kernel, which is determined in turn by the indirect utility function J (t) in (2.4) and (2.5). It suggests that the risk free rate contains many investing characteristics of the representative agent.

Define two functions f and g such that:

$$f = \frac{\lambda_1 - \rho\lambda_2/\sqrt{1-\rho^2}}{\sigma_1}, g = \frac{\lambda_2}{\sqrt{1-\rho^2}}$$

Notice that from condition (2) in proposition 1, simple algebra shows that $f=-\frac{G_s}{G}, g=-\frac{G_\varsigma}{G}$, so there exists a function y(.) such that

(2.15)
$$\frac{G_t}{G}(t,s,k)=-y(t)-\int_{s_0}^{s} f_t(t,\xi,k)d\xi-\int_{k_0}^{k} g_t(t,s,v)dv$$

Now, in terms of functions f and g, (2.14) can be rewritten as:
(2.16)
$$\frac{G_t}{G}-(\mu_1-D)f-\mu_2 g+\frac{1}{2}\sigma_1^2\left(f^2-f_s\right)+\frac{1}{2}\sigma_2^2\left(g^2-g_\varsigma\right)+\rho\sigma_1\sigma_2(f_\varsigma-fg)+r=0$$

Plugging (2.15) to (2.16), using the martingale restriction condition $\mu_1-D-rS=\lambda_1\sigma_1$ and differentiating the resulting equation with respect to s and k, we get two PDEs that must be satisfied by equilibrium viable MPRs, as shown in

Proposition 2*: In an economy as specified and the functions f and g defined above, in equilibrium f and g satisfies the following partial differential equations:*

(2.17)

$$\begin{aligned}&f_t+[r+(\rho\sigma_1(1+\sigma_2))_s g]f+\sigma_1(\sigma_1)_s f^2+[rs+\sigma_1^2 f+\rho\sigma_1(1+\sigma_2)g+\sigma_1(\sigma_1)_s]f_s\\&+[\mu_2-\sigma_2^2 g+\rho\sigma_1(1+\sigma_2)f+(\rho\sigma_1\sigma_2)_s]f_k+\frac{1}{2}\sigma_1^2 f_{ss}+\frac{1}{2}\sigma_2^2 f_{kk}+\rho\sigma_1\sigma_2 f_{sk}\\&+(\mu_2)_s g+\sigma_2(\sigma_2)_s(g_k-g^2)=0\end{aligned}$$

(2.18)
$$\begin{aligned}&g_t+[(\mu_2)_k+(\rho\sigma_1(1+\sigma_2))_k f]g-\sigma_2(\sigma_2)_k g^2+[rs+\rho\sigma_1(1+\sigma_2)g+\sigma_1^2 f+(\rho\sigma_1\sigma_2)_k]g_s\\&+[\mu_2-\sigma_2^2 g+\rho\sigma_1(1+\sigma_2)f+\sigma_2(\sigma_2)_k]g_k+\frac{1}{2}\sigma_1^2 g_{ss}+\frac{1}{2}\sigma_2^2 g_{kk}+\rho\sigma_1\sigma_2 g_{sk}\\&+\sigma_1(\sigma_1)_k(f^2+f_s)=0\end{aligned}$$

Note that (2.17) and (2.18) are similar to equations (3.13) and (3.14) in Pham and Touzi (1996). The difference is that in our model the dividend policy is treated exogenous hence we do not have the dividend terms as in Pham and Touzi (1996).

Now define the market relative risk aversion coefficient $\gamma(t)=-e(t)J_{ee}(t)/J_e(t)$ [14]. Proposition 3 follows immediately:

[14] As pointed out by Constantinides (1990) it is inappropriate to define the coefficient in terms of changes of consumption in an intertemporal model. Detemple and Zapatero (1991) also give an

Proposition 3*: In an economy as specified above, in equilibrium*

$$(2.19) \quad \gamma(t)=\frac{\lambda_1(t)-\rho(t)\lambda_2(t)/\sqrt{1-\rho(t)^2}}{\sigma_1(t)/S(t)}$$

Proof: the definition of MRA combined with condition (1) in proposition 1 and market clearance condition $e_t^*=S_t$ gives an expression for the instantaneous relative risk aversion $\gamma_t=-S(t)G_S(t)/G(t)$. Using condition (2) in proposition 1, we have the desired relation.

3. Discussion

3.1. Time-separable utility function

It can be shown that if we assume a usual time-separable utility function then we get the same result as in condition (2). To see this, first note that condition (A.1) in (2.5) now becomes $u_c(c^*,t)=J_e$. Since the market is complete, it is well known (Cox and Huang (1989)) that the stochastic control problem (2.3) can be transformed to a static optimization problem as follows:

$$\max_{\{c_t\}} E[\int_0^T u(t,c_t,x_t)dt+U(e_T)]$$

$$s.t.\ E\left[\int_0^T [G(t)c(t)+G(T)e(T)]dt \middle| I_t\right]\le S_0$$

Its first order condition is $u_c(t)=\varepsilon G(t)$, which gives our desired result $\frac{J_e(t,e_t,V_t)}{J_e(0)}=G(t,S_t,V_t)$. Condition (2) can be obtained similarly by dropping the terms involving habit level x_t.

3.2. Endowment economy

Consider an endowment economy where investors are endowed a total supply of one share of risky asset S with dividend rate δ(t, S, ζ) as in Pham and Touzi (1996).

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expression for the risk aversion coefficient in footnote 7. But the coefficient is defined as an atemporal gamble restricted to consumption, not capital, at date t.

The representative agent adopts a habit formation utility function as described above[15]. The equilibrium conditions are: (1) investors consume all dividends at any instant time t; (2) the market clears with zero net supply for riskless bond and other contingent claims, with one unit supply of risky asset S; (3) the total wealth e(t) = S(t). We claim that in this economic setting our main result (2.19) still holds. The proof is almost identical to before except that we now replace the consumption by dividends in all equilibrium conditions, so we omit it here for brevity. This shows that (2.19) is indeed quite robust in the sense that it is independent of the underlying economy and utility assumptions.

3.3. Option-implied MRA

3.3.1. Literature Review

Studies on implied MRA from options have a long history. Sprenkel (1967) extracted MRA using his warrant pricing formula involving a risk aversion parameter. Bartunek and Chowdhury (1997) are the first to estimate MRA from option prices. Assuming a power utility function they estimate the coefficient of constant relative risk aversion (CRRA). Adding exponential utility functions Bliss and Panigirtzoglou (2004) obtain estimates of MRA using British FTSE 100 and S&P 500 index options. Kang and Kim (2006) extend the analysis by assuming wider classes including HARA, log plus power and linear plus exponential utility functions. Despite the facility of implementing the models, however, results derived using the preference-based approach may be misleading because the models employed above either fail to incorporate stochastic volatility which should be included for any reasonable option pricing model, or depend heavily on utility function forms. Attempts in the literature to deal with these two issues include Benth, Groth and Lindberg (2009) who estimate MRA with a utility indifference approach assuming an exponential utility. They find a smiling MRA across strikes and time to maturities, concluding that certain crash risk is not captured by stochastic variance. Blackburn (2008) extracts the MRA and inter-temporal substitution assuming a non-time-separable Epstein-Zin type utility function. He finds reasonable estimates of the risk aversion parameter and claims that changes to risk aversion are closely related to changes of market risk premium, which is consistent with the theoretical results we derived. In contrast, our approach not only takes into account stochastic volatility but also accommodates any form of utility function, whether time separable von Neumann Morgenstern or habit persistence.

[15] We noted earlier that an endowment economy with habit formation is problematic for examining asset pricing as pointed out by Chapman (1998). Here we consider this case only to illustrate the robustness of our results.

MRA can also be extracted as the distance between the option implied risk neutral density and market subjective density of the underlying security returns assuming a pure exchange economy (Jackwerth (2000), Ait-Sahalia and Lo (2000), Rosenberg and Engel (2002), Bliss and Panigirtzoglou (2002), Perignon and Villa (2002)).That the empirical pricing kernel derived is locally increasing against wealth levels (or equivalently the implied MRA is locally negative) is termed the pricing kernel puzzle[16]. Using different estimation methods for pricing kernels, Ait-Sahalia and Lo (2000), Jackwerth (2000), and Rosenberg and Engle (2002) report that implied MRA exhibits strong U-shape across S&P 500 index values and the current forward prices. However, Singleton (2006) doubts that these findings may not be robust due to the simplified assumptions made about the underlying economy and the dimensionality of the state vector. Ait-Sahalia and Lo (2000) also acknowledge that for more complex economic models the distance between risk neutral and subjective probabilities may be misleading as a measurement of investors' aggregated risk preferences. A notable difference exists between the MRA in these studies and the one in this paper. The former is a function of the spot price at maturity date T; the latter is a function of the spot price at an instant time t. It is this feature that allows us to construct a time series of risk aversion index for the US market.

3.3. 2. Relation with Black-Scholes risk aversion

(2.19) implies that in the case of non-zero correlation the MRA can be written as the Black-Scholes risk aversion plus an extra term:

(3.1) $$\gamma_t = \gamma_{BS} - \frac{\rho\lambda_2}{\sigma_1 / (S\sqrt{1-\rho^2})}$$

This extra term is similar to the Black-Scholes term except that the volatility in the denominator is adjusted by the correlation. Given the empirical evidence that the correlation coefficient is generally negative for market index, whether the MRA is larger or smaller than its benchmark level is determined purely by the sign of MPOR. As pointed out in Lewis (2000) the sign of MPOR depends on whether the contingent claim in the zero-beta hedging portfolio is a call or put option[17]. A zero MPOR

[16] Jackwerth (2004) provides a good survey on this topic. Since then, Brown and Jackwerth (2004), Ziegler (2007), Chabi-Yo, Garcia and Renault (2007) offer possible explanations trying to reconcile the puzzle.

[17] To see this, consider a self-financing portfolio $X_t = \omega_t F_t + S_t$. The portfolio is zero-beta if dx $\circ$ ds=0. Using Ito's Lemma and solve for ω, we have $\omega = -(F_s + \rho\sigma_2 F_\varsigma / (S\sigma_1))$. Then the volatility of the hedging portfolio can be shown to be proportional to the "delta-vega" hedged portfolio value $(F - SF_s - \rho\sigma_2 F_\varsigma / \sigma_1)$. Note that the vega is multiplied by the correlation coefficient since the portfolio only aims to hedge away the risk associated with the underlying. The sign of this "delta-vega"

corresponds to a logrithmatic utility form as shown in Pham and Touzi (1996). If the MPOR is positive, indicating the market rewards investors for taking the risk due to orthogonal volatility randomness, then intuitively market players as a group should exhibit higher risk aversion than without this randomness. And a negative MPOR implies a decrease of the risk aversion relative to the Black-Scholes case. In addition, the risk aversion is inversely related to volatility. Volatility itself is mean reverting. If the current volatility level is low market players tend to expect higher future volatilities and hence would exhibit higher risk aversion. On the contrary if the current volatility is high, players would expect lower volatility in the future, hence lower risk aversion. Notice also that in (3.1) the MRA cannot be written as a function of market price of volatility risk (MPVR) only, suggesting the importance of separating MPAR and MPOR when MPOR is in presence.

3.3.3. Market risk aversion and volatility risk premium

We can relate the two through equation (3.1): $\gamma = (\lambda_1 - \upsilon\rho)/(1-\rho^2)\sigma_1$ where υ denotes the MPVR. Given the consensus that the stock return and volatility process are negatively correlated, the market representative agent will exhibit risk aversion if and only if the MPVR is greater than λ_1 / ρ and risk loving if and only the MPVR is less than λ_1 / ρ. In other words, our result show that the more negative the MPVR is, i.e. the more payment the agent makes to the market for taking volatility risks, the more risk loving the representative agent exhibits. The other way to write the above formula is to express the excess stock return in terms of the risk aversion and MPVR: $\lambda_1 = \gamma(1-\rho^2)\sigma_1 + \upsilon\rho$. So the excess return can be explained by two parts: one associated with the stock volatility itself, the other with the additional source of risk, the stochastic volatility, as captured by the variance risk. Notice that this expression is very similar to expression (11) in Bollerslev et al. (2009) where they consider a pure exchange economy with Epstein-Zin-Weil recursive preferences.

3.4. Link to Garman (1976)

It is well known that in general the risk neutral valuation implies the following pricing partial differential equation (PDE)

$$\text{(3.2)} \quad \frac{\partial V}{\partial t} + \frac{1}{2}(\nabla^2 V) \otimes \Sigma + \nabla V \bullet [\vec{\mu} - D\lambda] = rV$$

---

hedged portfolio value depends on the type of contingent claims. In empirical works one can calculate these Greeks to help judge the sign of MPRO for call and put options.

for multivariate pure diffusion models, where ∇V is the gradient of contingent claim value V with respect to the state variable vector $\vec{S}$, $\nabla^2 V$ the corresponding Hessian matrix, $\otimes$ a dyadic matrix operator, $\Sigma(t,T;\vec{S})$ the volatility covariance matrix. And condition (1) in Proposition 1 and equation (2.14) can be re-written in matrix form: $\vec{\lambda}_t = -D_t^T \nabla_{T\to t}{}' G$ and $\nabla_{T\to t} G \bullet \vec{\mu} + \frac{\partial G}{\partial t} + \frac{1}{2}\nabla_{T\to t}{}^2 G \otimes \Sigma = -r$. Plug these two conditions to (3.2) and recall $DD^T = \Sigma$ we have

(3.3)

$$\frac{\partial V}{\partial t} + \frac{1}{2}(\nabla^2 V)\otimes\Sigma + \nabla V \bullet [\vec{\mu} + \Sigma \bullet \nabla_{T\to t} G'] + [\nabla_{T\to t} G \bullet \vec{\mu} + \frac{\partial G}{\partial t} + \frac{1}{2}\nabla_{T\to t}{}^2 G \otimes \Sigma] V = 0$$

Notice that (3.3) is exactly the fundamental pricing differential equation under the multivariate case provided by Garman (1976). There it is derived through an inter-temporal parity rule which makes no assumptions on the market pricing kernel and in particular risk neutrality. Hence (3.3) is more general than the risk neutral valuation formula (3.2).

3.5. Examples

3.5.1. GBm and ABm

In the benchmark Black-Scholes world the underlying stock price process follows a Geometric Brownian motion (GBm). The pricing kernel function $G(t, S_t; T, S_T) = A(t,T)(S_T / S_t)^{g(t)}$ where $A(t,T)$ is the discount factor, $g(t) = -\lambda(t)/\sigma(t)$. This implies that the marginal utility function is $U'(S) = S^g$. By (1) and the equality $G(t) = U_c(t) = J_e(t)$ as established in the proof of proposition 1, the representative agent's utility function is then recovered to be $U(S) = S^{1+g}/(1+g)$ which is a power utility with constant relative risk aversion (CRRA) $-g = \lambda/\sigma$ as given in (2.14) with $\lambda_2 = 0$. Similarly, in the arithmetic Brownian motion (ABm) case it is known that the pricing kernel G takes the form of

$$G(t, S_t; T, S_T) = \exp\{-(\lambda/\sigma)(S_T - S_t)\}\exp\{-(r + \lambda\mu/\sigma + 1/2\lambda^2)(T-t)\}$$

A little algebra then shows this is an exponential utility function with constant absolute risk aversion (CARA) λ/σ, which again is consistent with our derived risk aversion (2.19).

3.5.2. Hull and White (1987)

Two major assumptions, namely zero correlation and zero volatility risk premium are made in Hull and White (1987) stochastic volatility model. We first consider the case where only volatility risk premium is assumed to be zero. Under this assumption, $\rho\lambda_1 + \sqrt{1-\rho^2}\lambda_2 = 0$, then by (2.19) we have $\gamma_{HW,\rho\neq 0} = \left(\frac{1}{1-\rho^2}\right)\frac{\lambda_1}{\sqrt{V}}$. If the correlation is zero also then the Hull-White risk aversion is exactly the same as in the Black-Scholes case. The intuition is that investors preference should not relate to stochastic volatility under Hull and White because the underlying process is independent of the volatility process.

3.5.3. Bollerslev et al. (2010)

We use this example to demonstrate that the relation between volatility risk premium and relative risk aversion in Bollerslev et al. (2010) effectively ignores the MPOR. The VRP is $\xi\sqrt{V}\left(\rho\lambda_1 + \sqrt{1-\rho^2}\lambda_2\right)$ where ξ is the volatility of variance, dependence on time t is omitted for brevity. As in Heston (1993), Bollerslev et al. (2010) assume a linear VRP with respect to the variance such that we can write $\xi\left(\rho\lambda_1 + \sqrt{1-\rho^2}\lambda_2\right) = \lambda\sqrt{V}$. Let the MPOR be zero and use (2.19) to derive relative risk aversion $\gamma = \lambda / \rho\xi$, which is exactly the relation used for their empirical study of the risk aversion.

4. Data and calibration

In this section we conduct a model calibration to extract empirical MPAR and MPOR and hence MRA through (2.19). The stochastic volatility model we consider is the Stein and Stein (1991) as described below. The volatility process follows a Ornstein-Uhlenbeck process: $d\sigma_1(t) = -a\sigma_1(t)dt + bdZ_t$, then Ito's lemma shows that the variance process ς (t) follows the Feller or CIR process with drift term $\alpha(m-\varsigma_t)$, the volatility of variance term $\sigma_2(t) = \beta(t)\sqrt{\varsigma(t)}$, with the constraint on parameters $\beta^2 = 4\alpha m$. The pricing PDE for the model is:

(4.1)

$$\frac{\partial V}{\partial t} + \frac{1}{2}V_{ss}\varsigma_t S_t^2 + \frac{1}{2}V_{yy}\beta^2\varsigma_t + \rho\beta\varsigma_t S_t V_{s\varsigma} + rV_s S_t + V_\varsigma\left[\alpha(m-\varsigma_t) - \beta\sqrt{\varsigma_t}(\rho\lambda_1 + \sqrt{1-\rho^2}\lambda_2)\right] = rV$$

We derive an analytic solution following the procedures described in Heston (1993) (see Appendix B for details). Our solution is highly accurate and fast compared with other valuation methods such as Heston model and Finite Difference Method, as shown in Table 1.

<Insert Table 1 here>

Model parameters are calibrated using OptionMetrics daily prices on the S&P 500 index call and put options traded on CME from September 2nd, 2008 to October 22nd, 2008. We use the average of bid and ask prices if neither is zero, otherwise we delete the data. We also delete all those with volume zero. Unlike many previous studies, we keep those options deep in or out of the money as we believe those also reflect market players' risk attitude[18].

Our model is based on the option in the basis. However, in practice it is difficult to check each option and find the one(s) that satisfies the conditions in Hugonnier et al. (2009). To deal with this realistic issue, we take two extreme views. In one case, we require the use of all available options as inputs for model calibration to find out a single MPOR. This is consistent with the claim in Buraschi and Jackwerth (2001) that all traded options are needed to span the market. In the other case, we assume that each traded put/call option completes the market. We admit that this is too strong an assumption to make. Nevertheless, this assumption is probably not that way off as one would think. The only model parameter that can change in this setting is the MPOR. All other parameters, including the MPAR, cannot possibly change as all options are written on the same underlying security. Intuitively, investors trade options with different maturities and strikes because they have different risk preferences.

Figure 1 presents a time series of MPOR, MRA, MPVR and MPAR extracted from daily option prices around the Lehman Brothers bankruptcy, namely from September 2nd 2008 to October 22nd 2008[19]. The MPAR or the Sharpe ratio, mostly positive and between -0.05 and 0.30, has a slightly decreasing trend. The MPOR is mostly negative and has no clear trend. The MPVR, corresponding to the market compensation for taking stochastic volatility risk, is also negative, consistent with the current literature, and between 0.2 and -0.65. The behavior of MPVR is mostly driven by the MPOR, which demonstrates that understanding the often neglected MPOR is

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[18] Theory in Cvitanic, Jouini, Malamud and Napp (2009) predicts that in the options market the market pricing kernel should exhibit average behavior for a certain range of strikes around at the money and is dominated by individuals outside this range.

[19] We could have used longer time series of data, but we'd rather leave it to another research project since the focus of this paper is on the theoretical derivation of, not empirical behavior of market risk aversion.

quite important for understanding the volatility risk premium. The MRA is between -0.5 and 2, with most of its values falling within the positive zone. Again this is consistent with current literature on option-implied MRA. The Figure clearly shows that the MRA is driven by the MPAR.

< Insert Figure 1 here >

Figure 2 and 3 shows cross sectional MPOR and MRA for calls and puts respectively on September 15th, 2008, the date on which Lehman Brothers went bankrupt. For both puts and calls and for all contracts, the MRA and MPOR increase across the strikes. This implies that investors trading deep out of the money calls and deep in the money puts are more risk averse than those trading deep in the money calls and deep out of the money puts. From a time series perspective, for calls the level of MRA increases but the variation of the MRA decreases with time to maturity. Investors trading calls with more than two years of time to maturity share the same level of risk aversion. Investors trading calls within two weeks before maturity vary very much in their risk preference. For puts, we observe almost the same pattern for contracts with strikes less than $1,275. The level of MRA increases and variation decreases. The completely opposite pattern is observed for those beyond the strike of $1,275. Contracts with less time to maturity are more risk averse and the variations for all contracts are much smaller.

< Insert Figure 2 and Figure 3 here >

5. Conclusion

In this paper we assume a production economy where state variables follow a stochastic volatility model. Under mild restrictions the market is dynamically complete and equilibrium exists. We derive a relationship between market prices of risks and risk aversion when the economy is in equilibrium. The relation is robust because it also holds under an endowment economy and no matter what risk preference, habit formation or regular von-Neumann-Morgenstern, the representative agent has. We demonstrate how to use the relation to empirically construct a time series of and cross sectional market risk aversion.

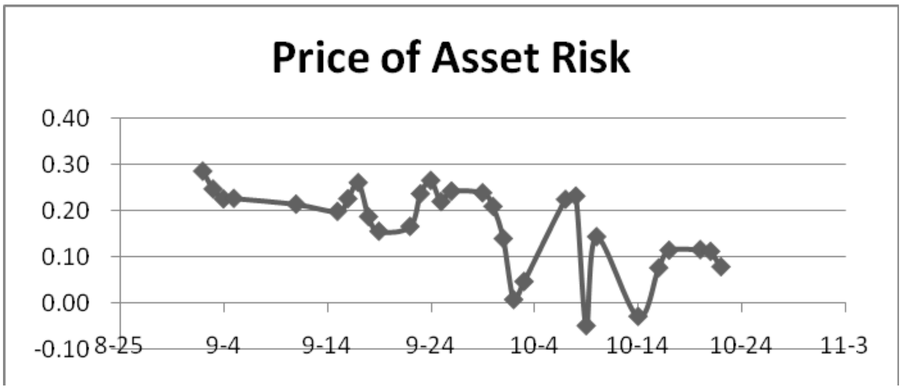
Future theoretical research can consider the validity of the relation under jump-diffusion models in which case a new market price of jump risk will be introduced. Another line of research could be on how to model the empirical behavior of MRA and include it into the option pricing formula, as in earlier works such as Sprenkel (1961) who explicitly incorporates a risk aversion parameter in his warrant pricing formula. On the empirical application side, a longer time series of market risk aversion can be extracted from options market. Whether this time series contain useful information about market sentiment and how the series relate to macroeconomic variables will be an interesting topic.

Tables and Figures

Table 1 – Pricing method comparisons.

interest rate	r	0	0.05369	0.1
dividend	d	0	0	0
strike	K	100	1050	123.4
maturity	T	0.5	1	1
vol of volatility	σ	0.1	0.1136	0.15
mean reversion rate	κ	2	3.2489	1.98894
mean reversion level	θ	0.01	0.77422	0.01188
corr. coefficient	ρ	0	-0.3372	-0.9
spot price	S	100	1268.21	123.4
volatility	ν	0.01	0.03	0.02
market price risk	λ	0	0	0
Heston		2.79116	481.28	13.8571
Finite Difference		2.7938	481.563	13.8364
Stein and Stein		2.79115	481.258	13.864

Figure 1 – Time series of market risk aversion, price of volatility risk, price of asset risk and price of orthogonal risk from September 4th, 2008 to October 22nd, 2008. The values are extracted using all traded calls and puts on each day.



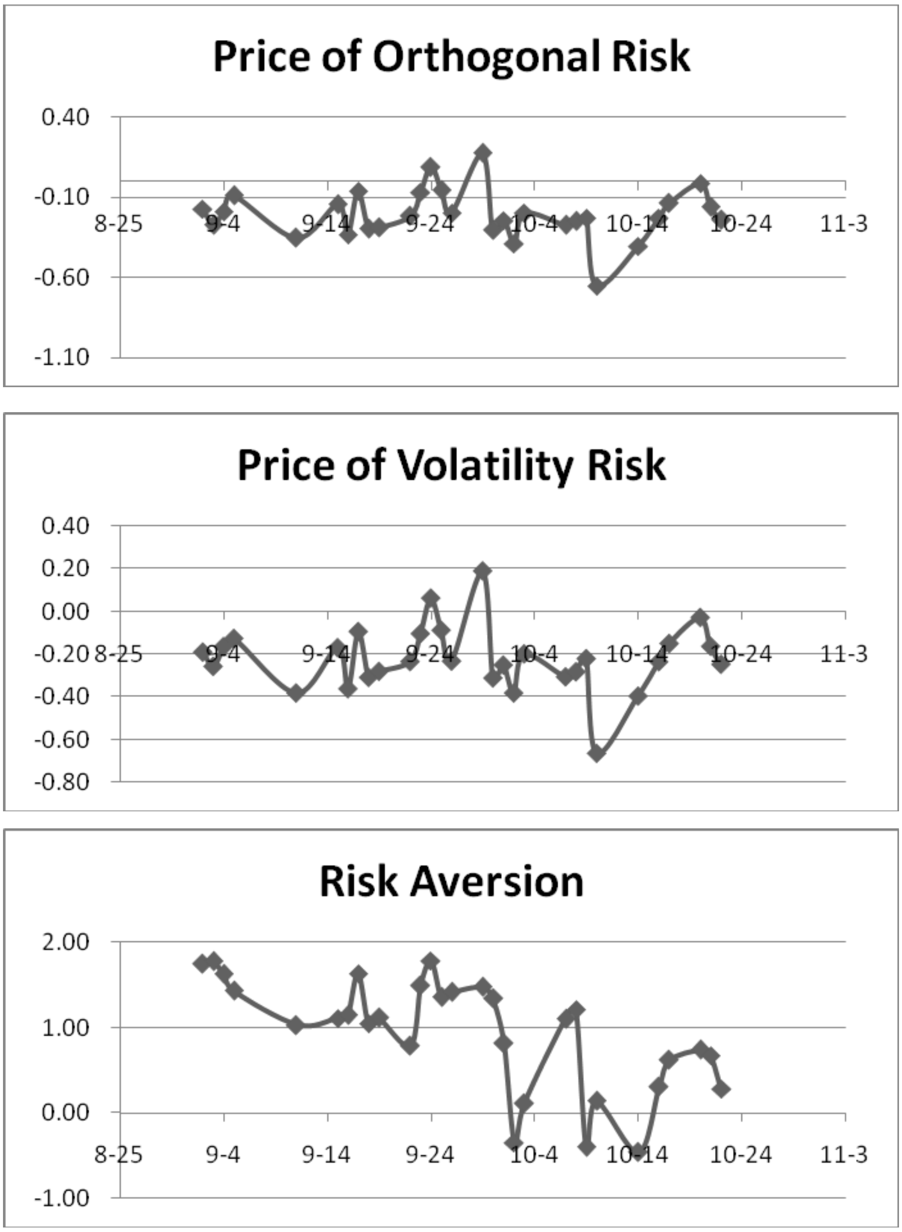
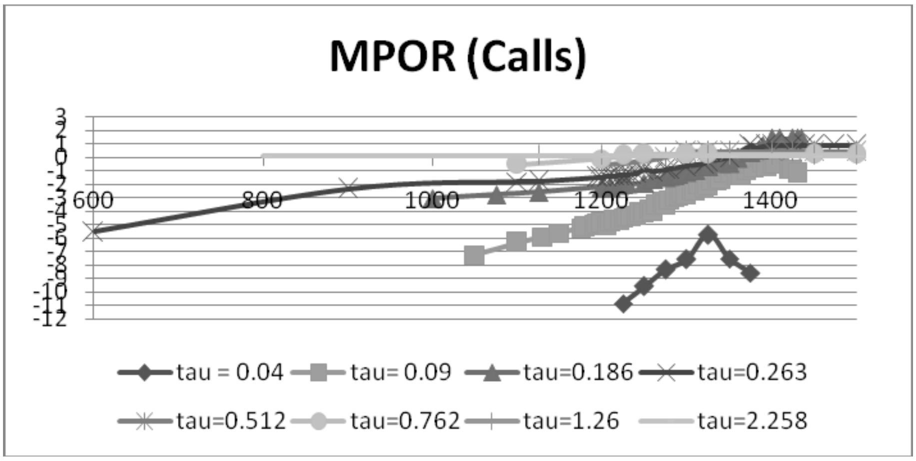


Figure 2 – Cross sectional market risk aversion (MRA) and price of orthogonal risk (MPOR) against the strike price for call options on September 15^{th}, 2008, tau is the time to maturity. The values are extracted using each of 167 calls traded on that date.



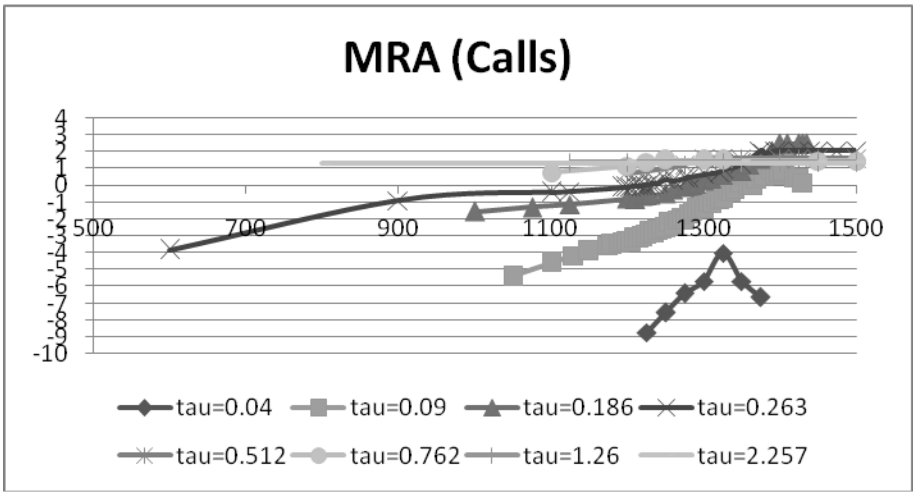
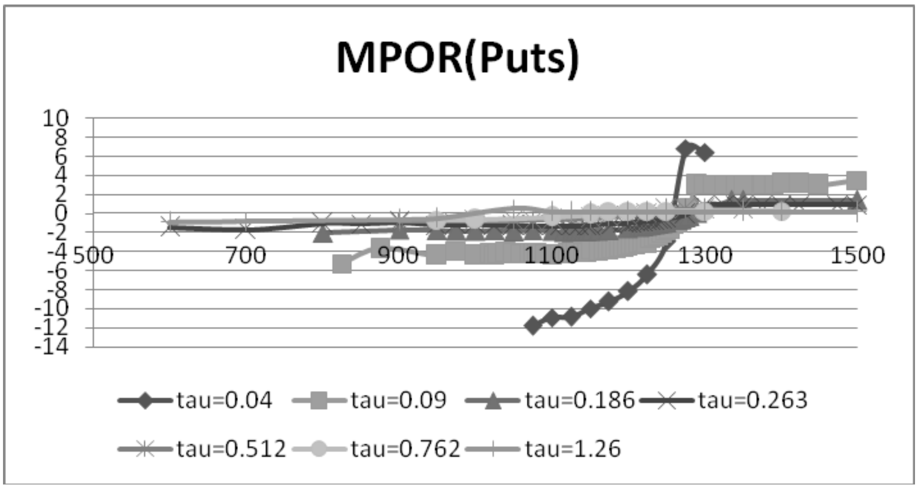
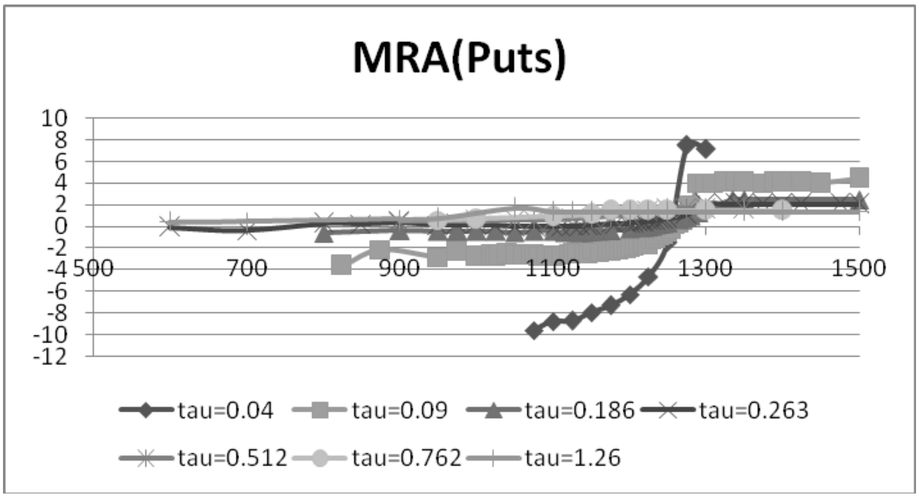


Figure 3 – Cross sectional market risk aversion (MRA) and price of orthogonal risk (MPOR) against the strike price for put options on September 15th, 2008, tau is the time to maturity. The values are extracted using each of the 194 puts traded on that date.





Appendix A – Proof of Proposition 1

Let $J\left(t, e_t, S_t, V_t, x_t\right)$ be the value function for problem (2.3). Then Hamilton-Jacobi-Bellman equation gives:

$$0=\max_{(\alpha,\beta)}\left[u(t,c_t,x_t)+E_t\left(\frac{dJ}{dt}\right)\right]$$

Apply Ito's lemma to $J\left(t,e_t,S_t,V_t,x_t\right)$ and plug in the stochastic differential equations for the underlying asset, the put option and the wealth, we have:

(2.4)

$$0=u(c,x)+J_t+aJ_e+(\mu_1-D)J_S+\mu_3 J_V+J_x(k_2c-k_1x)+\frac{1}{2}J_{ee}(b^2+c^2)+\frac{1}{2}J_{ss}\sigma_1^2+\frac{1}{2}J_{VV}F^2+b\sigma_1 J_{Se}$$
$$+(\beta\frac{F^2}{V}+\alpha\frac{\sigma_1 F}{S}\rho^{'})J_{Ve}$$

First order conditions with respect to c, α and β are:

(2.5)

$$0=u_c(c^*,t)+k_2J_x-J_e \qquad (A.1)$$
$$0=\left(\frac{\mu_1-D}{S}-r\right)J_e+\frac{\sigma_1}{S}\left(\frac{\alpha\sigma_1}{S}+\rho^{'}\frac{\beta F}{V}\right)J_{ee}+\frac{\sigma_1^2}{S}J_{Se}+\frac{\sigma_1 F\rho^{'}}{S}J_{Ve} \qquad (A.2)$$
$$0=\left(\frac{\mu_3}{V}-r\right)J_e+\left[\left(\frac{\alpha\sigma_1}{S}+\rho^{'}\frac{\beta F}{V}\right)\frac{\rho^{'}F}{V}+\left(1-\rho^{'2}\right)\frac{\beta F^2}{V^2}\right]J_{ee}+\rho^{'}\frac{\sigma_1 F}{V}J_{Se}+\frac{F^2}{V}J_{Ve} \qquad (A.3)$$

Now, differentiating (2.4) with respect to e we have:

(2.6)

$$0=J_{te}+rJ_e+aJ_{ee}+(\mu_1-D)J_{Se}+\mu_3J_{Ve}+J_{xe}(k_2c-k_1x)+\frac{1}{2}J_{eee}(b^2+c^2)+\frac{1}{2}J_{sse}\sigma_1^2+\frac{1}{2}J_{VVe}F^2+b\sigma_1J_{See}$$
$$+(\beta\frac{F^2}{V}+\alpha\frac{\sigma_1F}{S}\rho^{'})J_{Vee}+\sigma_1F\rho^{'}J_{SVe}$$

This suggests that dJ_e has a drift term $-rJ_e dt$ and its diffusion terms are[20]

$bJ_{ee}+\sigma_1J_{Se}+\rho^{'}FJ_{Ve}$ for dW_{1t} and $cJ_{ee}+\sqrt{1-\rho^{'2}}FJ_{Ve}$ for dW_{2t}. That is,

$$dJ_e=-rJ_e dt+(bJ_{ee}+\sigma_1J_{Se}+\rho^{'}FJ_{Ve})dW_{1t}+(cJ_{ee}+\sqrt{1-\rho^{'2}}FJ_{Ve})dW_{2t}$$

---

[20] Apply Ito's Lemma to the function $e^{rt}J_e(t,e_t,S_t,V_t,x_t)$ and use (2.6) for the drift term.

We claim that the above can be rewritten as:

(2.7) $dJ_e = -rJ_e dt - \lambda_1 J_e dW_{1t} - \lambda_2 J_e dW_{2t}$

To see this, recall that the discounted underlying asset and discounted put option value are both martingales under the risk-neutral measure defined through market prices of risks. For the former, we have

(2.8) $\lambda_{1t} = \dfrac{\mu_{1t} - D_t - rS_t}{\sigma_{1t}}$

For the latter we have

(2.9) $\dfrac{\mu_{3t} - rV_t}{F_t} = \rho^{'} \lambda_1 + \sqrt{1-\rho^{'2}}\lambda_2$

Note that the left hand is the market price of option risk, while the right hand side is a linear combination of the MPAR and the MPOR. Hence (2.9) provides a nice relationship between the three market prices of risks.

Now we plug (2.8) into equation (A.2) in (2.5) to get:

(2.10) $-\lambda_1 J_e = bJ_{ee} + \sigma_1 J_{Se} + \rho^{'} F J_{Ve}$

On the other hand, we write equation (A.3) in (2.5) by adding and subtracting the same term as:

$$(\mu_3 - rV)J_e + bF\rho^{'} J_{ee} + \left(1-\rho^{'2}\right)\frac{\beta F^2}{V} J_{ee} + \sigma_1 F \rho^{'} J_{Se} + \rho^{'2} F^2 J_{Ve} - \rho^{'2} F^2 J_{Ve} + F^2 J_{Ve} = 0$$

Using (2.9) the above is equivalent to

$$(\mu_3 - rV - \lambda_1 F\rho^{'})J_e + \left(1-\rho^{'2}\right)\frac{\beta F^2}{V} J_{ee} + F^2(1-\rho^{'2})J_{Ve} = 0$$

Notice that the bracket term in the J_e term equals $F\sqrt{1-\rho^{'2}}\lambda_2$ by (2.9), plug in and rearrange we have

(2.11) $-\lambda_2 J_e = cJ_{ee} + F\sqrt{1-\rho^{'2}} J_{Ve}$

(2.10) and (2.11) together imply (2.7), which is equivalent to $J_e(t,e_t,S_t,V_t,x_t) = J_e(0)\exp\left\{-rt - \int_0^t \lambda_{1s}dW_{1s} - \frac{1}{2}\int_0^t \lambda^2{}_{1s}ds - \int_0^t \lambda_{2s}dW_{2s} - \frac{1}{2}\int_0^t \lambda^2{}_{2s}ds\right\}$ which implies $\frac{J_e(t,e_t,S_t,V_t,x_t)}{J_e(0)} = G(t,S_t,V_t,x_t)$, hence there exists a (indirect) utility function that supports the pricing kernel. The terminal condition comes from the fact that at the terminal period the agent will not care about the volatility and just consume the underlying asset.

To derive condition (2) in the proposition, we define a function H such that

$$H(t,S_t,\varsigma_t,x_t) = E\left[e^{r(T-t)}U^{'}(S_T)\middle|S_t,\varsigma_t,x_t\right] = J_e(0)E\left[e^{r(T-t)}G(S_T)\middle|S_t,\varsigma_t,x_t\right] = J_e(0)G(t,S_t,\varsigma_t,x_t)$$

The last equivalence comes from the fact that $e^{rt}G(t)$ is a martingale. Now applying Ito's lemma to $H(t,S_t,\varsigma_t,x_t)$ to get:

(2.12)

$$dH(t) = (H_t + (\mu_1 - D)H_s + \mu_2 H_\varsigma + (k_2 c - k_1 x)H_x + \frac{1}{2}H_{ss}\sigma_1^2 + H_{s\varsigma}\rho\sigma_1\sigma_2 + \frac{1}{2}H_{\varsigma\varsigma}\sigma_2^2)dt$$
$$+(\sigma_1 H_s + \rho\sigma_2 H_\varsigma)dW_{1t} + \sqrt{1-\rho^2}\sigma_2 H_\varsigma dW_{2t}$$

By definition of the pricing kernel, we have

(2.13)

$$dG(t) = -rGdt - \lambda_1 GdW_{1t} - \lambda_2 GdW_{2t}$$

Now comparing the volatility terms in (2.12) and (2.13) gives the desired result. Equilibrium consumption can then be obtained from (A.1) in (2.5) by first solving the indirect utility function J(e).

Appendix B – Closed Form Solution to Stein and Stein (1991)

Heston (1993) derived a closed form solution for a stochastic volatility model by assuming that the risk premium term is proportional to the variance $v(t)$, i.e. $\lambda(S,v,t) = \lambda_t v$ using equilibrium results in Breeden (1979) and Cox, Ingersoll and Ross (1985). In this section, we will follow the techniques provided in Ingersoll (1986), Bates (1992) and Heston (1993) to derive a closed form solution to the Stein and Stein SVM in consideration. Under risk neutral valuation the market risk premium is proportional to the square root of variance, not the variance itself, i.e. $\lambda(S,\varsigma,t)=\sigma\sqrt{\varsigma(t)}(\rho\lambda_1+\sqrt{1-\rho^2}\lambda_2)$. As demonstrated in Pham and Touzi (1996), however, any assumption made on the form of risk premium translates into a specific assumption on the utility function. In this paper we want to relax the constraint on people's utility form as much as possible. In addition, one of our goal is to separate the λ_1 and λ_2, instead of treating them together as the market price of volatility risk as most current papers do. However, this inevitably introduces an extra term involving $\sqrt{v(t)}$ in the risk premium term. This change will require a new closed form solution for the pricing equation. Recall that the Stein and Stein (1991) SVM is of the following form:

$$dS(t)=\mu Sdt+\sqrt{v(t)}Sdz_1(t)$$
$$d\sqrt{v(t)}=-\beta\sqrt{v(t)}dt+\delta dz_2(t)$$

By Ito's lemma, the variance process is:

$$dv(t)=k(\theta-v(t))dt+\sigma\sqrt{v(t)}dz_2(t)$$

where $\sigma^2=4k\theta,\sigma=2\delta,k=2\beta$. Following Heston (1993) we assume the option price can be written in the form $C(S,v,t)=SP_1-KB(t,T)P_2$ where B (t,T) is the zero coupon bond price with maturity date T, P1 and P2 are "adjusted" probabilities with characteristic functions f1 and f2 respectively. Identical argument as in the appendix of Heston (1993) and a little algebra show that for our model the two characteristic functions follow the Fokker-Planck forward equations:

(1) $$\frac{1}{2}vf_{xx}+\rho\sigma vf_{xv}+\frac{1}{2}\sigma^2vf_{vv}+(r+u_jv)f_x+(a_j-b_jv-c_j\sqrt{v})f_v+f_t=0$$

where $u_j=(-1)^{j-1}\frac{1}{2},a_j=k\theta,b_j=k-(2-j)\rho\sigma,c_j=\sigma(\rho\lambda_1+\sqrt{1-\rho^2}\lambda_2),j=1,2$.

We guess that the characteristic functions take the following form:

(2) $$f(x,v,t)=\exp\left\{C(T-t)+D(T-t)v(t)+E(T-t)\sqrt{v(t)}+i\phi x\right\}$$

where $x=\log S$. Plugging (2) in (1) and collecting the terms involving $v(t)$, $\sqrt{v(t)}$ we have the following system of equations:

$$-\frac{1}{2}\phi^2+i\phi\rho\sigma D+\frac{1}{2}\sigma^2D^2+i\phi u_j-b_jD+D_t=0$$
$$i\phi r+a_jD+C_t-\frac{1}{2}c_jE+\frac{1}{8}\sigma^2E^2=0$$
$$\frac{1}{2}i\phi\rho\sigma E+\frac{1}{2}\sigma^2DE-\frac{1}{2}b_jE-c_jD+E_t=0$$

Solving the above PDEs, we have the expressions for D and E[21]:

$$D(\tau) = \frac{b - \rho\sigma\phi i + d}{\sigma^2}\left[\frac{1 - e^{d\tau}}{1 - g e^{d\tau}}\right]$$

$$E(\tau) = \frac{2c(b - \rho\sigma\phi i + d)}{d\sigma^2}\left[\frac{e^{d\tau} - 2e^{d\tau/2} + 1}{1 - g e^{d\tau}}\right]$$

where $d = \sqrt{\left(\rho\sigma\phi i - b\right)^2 - \sigma^2(2\mu\phi i - \phi^2)}, g = \dfrac{b - \rho\sigma\phi i + d}{b - \rho\sigma\phi i - d}$.

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[21] Solving for C is straightforward given D and E. The expression for C however is rather long. So to save space we omit it here but make it available upon request.

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