

Uncertainty Aversion and A Theory of Incomplete Contract

Chenghu Ma*

WISE, Xiamen University, Xiamen, China 361005

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Abstract

This paper is to provide a theoretical foundation of incomplete contract in an extensive game of multi-agent interaction. It aims to explain why rational agents may agree upon incomplete contracts even though it is costless to sign a complete one. It is argued that an incomplete contract creates strategic uncertainty. If agents' attitudes toward uncertainty are not neutral, then an incomplete contract as final solution can be the consequence of common knowledge of rationality. This paper assumes that all agents are uncertainty averse in a sense of Gilboa and Schmeidler (1989); and that agents can form coalitions as part of strategic play. All these are embedded into a newly proposed equilibrium solution concept for extensive form game of perfect information.

Key words: uncertainty aversion, strategic uncertainty, coalition-formation, stability and core-criterion.

JEL Classification: C70,C71,C72

1 Introduction

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This paper is to provide a theoretical foundation for incomplete contracts by maintaining the classical assumptions of individual rationality and common knowledge of rationality. We ask the following fundamental question: Do agents have incentive to sign incomplete contracts when it is costless to sign a complete one?

This paper distinguishes from the existing literature by threefold: First, contracts considered in this paper are treated as non-binding proposals. For any arbitrarily given contract, agents may or may not act as what prescribed in the contract — an agent chooses to or not to follow the contract depending on if what prescribed in the contract offered the best outcome to the agent, or not. As a necessary condition, an equilibrium contract must correspond to one that is to be honoured by all contractual parties. For expositional simplicity, we restrict our discussion with respect to the class of extensive games with perfect information. Since an action profile leads to a unique payoff for each parties and since actions are observable, a contract in this context is thus about action-taking at each decision node. A complete contract specifies a unique act for each decision node; and an incomplete contract may correspond to a set of acts at some decision nodes. It is in this sense, the newly proposed concept of equilibrium contract constitutes also a new game-theoretic concept for extensive games of perfect information, a concept differs from existing solution concepts with respect to this particular class of games.

The second distinguishing feature of this paper concerns about underlying behaviour assumptions on the agents. Similar to Ma (2000), we assume that all agents are uncertainty averse in a sense of Gilboa and Schmeidler (1989), and maximize the maximin expected utility. We shall see that the psychological aversion towards uncertainty, together with the nature of uncertainty carried through incomplete contracts, play a crucial role for agents to potentially agree upon some forms of incomplete contract.

Finally, the possibility of coalition-formation and formation of strategic partnership as driving forces towards formation of incomplete contract are also received special treatment in this paper.

1.1 Incomplete Contracts: A Critical Assessment

In literature, the existence of incomplete contracts has been regarded as the consequence of agents' inability to write a complete one. It was either due to the difficulty to verify if certain actions were really taken by the agents (Hart and Moore 1988), or due to the impossibility for the parties to specify all

relevant future contingencies (Maskin and Tirole 1999, and Mukerji 1998), or due to their limited ability to reason over the long term. The last situation is treated as a consequence of “bounded rationality”. This paper, on the other hands, aims to explore the strategic aspect of incomplete contract theory. We dig into the motivations for rational agents to agree upon incomplete contract even when a complete contract can be costlessly written. This, in turn, shed further lights on the nature of multi-agent interaction in general game-theoretic contexts.

Specifically, we wish to call for the attention to two important aspects of human behaviour, each contributing towards the incompleteness of contractual forms among the contractual parties. First, we concern about the possibility of coalition formation in multi-agent strategic interactions. For instance, once a sub-coalition is formed with some sorts of agreed plan of actions, which itself could be incomplete, to be reached among coalition members, this will leave the actions of agents outside the coalition to be unspecified. If this is the case, incompleteness of the contractual form constitutes the outcome for the grand coalition. Second, we point out that the existence of incomplete contract could be a direct consequence of agent’s psychological aversion towards uncertainty. It is well known that, uncertainty differ from risk in that the randomness of an uncertainty event may or may not be governed by a probability measure, and agents may not possess information on the probability assigned on the uncertain event even when it is associated with a probability. Since incomplete contracts create strategic uncertainty on the agents, the existence of incomplete contracts could be fully consistent with the common knowledge of uncertainty averse rationality to the economic agents.

Both coalition formation and strategic uncertainty associated with incompleteness of contract forms can be formally modelled. These are reflected from a newly proposed equilibrium concept for extensive form games of perfect information. The concept extends naturally to general strategic form games of possibly imperfect information.

1.2 Uncertainty and Uncertainty Averse Behavior Assumption

Uncertainty differs from risk in that randomness of an uncertain event cannot be adequately described by a probability measure. Such uncertainty is probably first discussed by Knight (1921), and is thus referred to as Knightian uncertainty. Tremendous advancements were made in theory of choice

under Knightian uncertainty, following the pioneering contributions made by Savage (1954) and many others (see Luo and Ma 1999 for a survey). Savage (1954) provided an axiomatic foundation for subjective probability and subjective expected utility theory for decision making. It suggests that, in facing Knightian uncertainty, a decision maker behaves as if there were a probability, the so-called subjective probability, associated with the randomness of uncertain events, and maximizes the underlying expected utility. This type of behavior assumption is referred to as “uncertainty neutrality”. Given its simplicity and its tractability, the uncertainty neutral behavior assumption is, however, found to be in contradict to experimental findings following Ellsberg (1961). In particular, experimental evidence suggests that, in making choices under uncertainty, people do exhibit some level of aversion toward uncertainty, and such choices are inconsistent with the existence of a subjective probability.

Consider a typical decision problem under uncertainty: Let X be a finite outcome space, Ω a finite state space, and $M(\Omega)$ the set of probability measures over Ω . Let F denote the act space that consists of all mappings from Ω to X . A preference relation $\succeq$ defined on the act space F is assumed to admit the following representation:¹ there exists an affine function $u : X \rightarrow R$, and a unique, non-empty, closed and convex correspondence $\Delta : F \rightarrow M(\Omega)$, such that, for all $f, g \in F$,

$$f \succeq g \iff \min_{p \in \Delta(f)} \int_{\Omega} u(f(\omega)) dp(\omega) \geq \min_{p \in \Delta(g)} \int_{\Omega} u(g(\omega)) dp(\omega). \quad (1)$$

The set $\Delta(f) \subseteq M(\Omega)$ is interpreted as the beliefs induced by the preference $\succeq$ and by act f . It suggests that, in making choices, a decision maker behaves as if they perceived a set of probability measures associated with uncertain events, and maximize the most conservative expected utility among the set of probabilities.

For the extreme case of complete vagueness toward the possible realization of the state of nature, if the decision maker exhibits maximum ambiguity aversion, the set of beliefs is to consist of all probability measures on Ω , *i.e.*, $\Delta = M(\Omega)$. In that case, the preference relation (1) reduces to

$$f \succeq g \iff \min_{\omega \in \Omega} u(f(\omega)) \geq \min_{\omega \in \Omega} u(g(\omega)). \quad (2)$$

¹This formulation is a generalization of Gilboa and Schmeidler (1989). Here, we do not require that $\Delta(f) = \Delta(g)$ for all acts $f, g \in F$. This is because different acts may lead to different contingent state spaces, thus different belief sets. Our formulation has some advantages in modelling decision making in sequential situations.

1.3 Incomplete Contracts and Endogenous Uncertainty

The uncertainty induced by an incomplete contract is referred to as **endogenous uncertainty** since it is created by the economic agents in contrast to the **exogenous uncertainty** which is about the realization of states of nature. The question is: why do agents want to create uncertainty when they are uncertainty averse? Intuitively, since agents are uncertainty averse, uncertainty created by one agent may affect other agents' optimal decisions. Therefore, uncertainty creations become relevant in multi-agent interactions. Indeed, examples can be constructed in showing that an incomplete contract can create a credible threat to the other parties; conversely, a complete, or less incomplete contract may serve as a credible promise to the others. It is shown in this paper that agents may indeed agree up on some equilibrium contracts that are incomplete. Moreover, examples are constructed in showing that incomplete contract may lead to Pareto dominating outcomes.

The literature on strategic uncertainty in game theory becomes active following the breakthrough of Gilboa and Schmeidler (1989) in static choice theory under uncertainty. Implications of uncertainty aversion behavior assumption on strategic form games are carried out by Klibanoff (1993), Lo (1996), Dow and Werlang (1994), and Eichberger and Kelsey (1994). Nevertheless, none of these papers deal with uncertainty creation in strategic plays. From this aspect, Ma (2000) and Luo and Ma (2001) are, to our knowledge, among the first to explore the implications of uncertainty creation in strategic interactions.

In fact, the theory of incomplete contracts developed in this paper is built upon earlier advancement made in Ma (2000) within the context of optimal contract. It is noted that, the various solution concepts proposed in Ma (2000) can be merely treated as outcomes of 'partial' equilibrium analysis because what was tackled in Ma (2000) concerns about how would agents act for an arbitrary given contract. The deeper issue on the formation of the equilibrium contract was not tackled in Ma (2000) nor in Luo and Ma (2001). This last ambition is accomplished in this paper. The Ξ -equilibrium contract as a solution concept proposed in this paper can be regarded as a coalition-proof and self-enforced equilibrium solution concept for extensive games of perfect information.

1.4 Stability and Core Criteria

Loosely speaking, we define **equilibrium contract** as a contract that satisfies the following two conditions:

- (a) Once the contract is proposed, agents have no incentive to deviate from it even though they have the freedom to do so;
- (b) By keeping (a) in mind, no coalitions can be formed in proposing a new agreement aiming to induce a final outcome that benefits all coalition members.²

Condition (a) is referred to as “stability” criterion; and condition (b) is referred to as “rationality” criterion. The rationality criterion proposed in this paper resembles the “core” condition in co-operative games, thus is also referred to as the core criterion.

The stability criterion resembles the “self-enforcement” property of Bernheim, Peleg and Whinston (hence BPW,1987).³ Once the equilibrium contract is proposed, it is expected to be optimally carried out even though it is not binding. It is in this sense we say that our equilibrium contract is self-enforcing.

The core criterion is similar to the coalition-proof condition proposed by Aumann (1959) and BPW (1987). It requires that, in equilibrium, no agents and coalitions can improve their payoffs by proposing a contract that is different from the original contract. In contrast to Aumann and BPW, we do not require the deviated contract to be stable, but instead agents will perceive the consequence of the actual play, namely the solution of the game as a consequence of the common knowledge of rationality and common knowledge of the newly proposed contract, and they will evaluate the perceived outcomes accordingly.

Indeed, the proposed equilibrium contract as a game-theoretical solution concept differs from the coalition-proof Nash equilibrium proposed by BPW, and from the strong Nash equilibrium proposed by Aumann — both

²The verifiability condition is not formulated as a constraint in the newly proposed equilibrium contract. This is because the possibility of actual actions deviating from the proposed actions in the “equilibrium” contract will necessarily contradict to the assumption of individual rationality.

³Thanks for Professor Bahskar for bringing to me the attention of the paper by BPW (1987).

constitute refinements to the Nash equilibrium.⁴ Moreover, in contrast to the coalition-proof Nash equilibrium, the equilibrium contract proposed in this paper does not depend on the procedure of coalition-formation in conducting a deviation (see footnote 2 in BPW 1987). An example can be constructed in showing that a game has an equilibrium contract, but it contains no coalition-proof Nash equilibrium and strong Nash equilibrium (see Example 3 below).

1.5 The Hold-Up Problem

To further illustrate the nature of conditions (a) and (b) imposed on equilibrium contracts, we consider the following hold-up problem documented in Hart (1995).⁵ An investment opportunity requires input from both M1 and M2 respectively at period 0 and period 1. If M1 invest i at $t = 0$, and if M2 decide to participate, which requires a fixed investment of c^* , a gross revenue $R(i)$ will be realized. It is assumed that $R(0) > c^*$, $R'(0) > 2$, $R'(\infty) < 1$ and that $R(\cdot)$ is increasing and concave. These assumptions are to ensure that there is always an incentive for M2 to invest, and that there exists a unique social optimal investment by M1, which is denoted as i^* . Here, i^* maximizes the total net profit $\pi(i) = R(i) - i - c^*$.

Without a contract, M1 has no incentive to invest at the social optimal quantity i^* . This is because M1's investment at $t = 0$ will become "sunk cost" at $t = 1$. The Nash bargaining between the two agents on dividing $R(i) - c^*$ is unavoidable before M2 making its decision to participate. Assuming that the two agents have equal bargaining power, each to receive $\frac{1}{2}(R(i) - c^*)$ as a result of bargaining. By anticipating this, as a rational agent, M1's optimal investment at $t = 0$ is to invest a quantity $\hat{i}$ that maximizes $\frac{1}{2}(R(\hat{i}) - c^*) - \hat{i}$. This, nevertheless, differ from the social optimal investment i^* . In fact, we must have $\hat{i} < i^*$, which corresponds to under-investment relative to the social optimal level.

Can the two parties instead agree on the following contract: M1 is to invest i^* and to receive a net profit of $\frac{1}{2}(R(\hat{i}) - c^*) - \hat{i} + \varepsilon$, ε is positive

⁴As pointed out by BPW, in defining the strong Nash equilibrium, the self-enforcement condition in forming a deviation was ignored. As a consequence, the strong Nash equilibrium almost never exists.

⁵We must emphasize the conceptual difference concerning the assumption made on the non-binding nature of contract employed in this paper, and the binding agreement implicitly assumed in Hart's book. Keeping this in mind, one shall not be surprised to find arguments and conclusions offered here to differ from what outlined throughout Hart's book.

and small, as what proposed in Hart (1995)? The allocation proposed in this contract improves both parties' payoffs in comparison to the outcome associated with investing at $\hat{i}$. It will, however, not be carried out! This contract is not credible to M1 in the sense that, if he invests i^* , he will actually receive $\frac{1}{2}(R(i^*) - c^*) - i^*$ which is less than $\frac{1}{2}(R(\hat{i}) - c^*) - \hat{i}$. This is because a round of renegotiation at $t = 1$ will be unavoidable given that M2 is rational. This contract violates condition (a) — the actual plays will for sure differ from what prescribed in the contract.

The fixed-fee type contract also covered in Hart (1995) will not resolve the hold-up problem. The contract suggests that: M1 invests i^* , and pays a fee $f + c^*$ to cover M2's cost. The logic works as follow: Suppose M1 follows the contract, and invests i^* at $t = 0$. The conditions under which neither agent deviates from the contract at $t = 1$ are:⁶

$$\pi(i^*) - f \geq \frac{\pi(i^*) + i^*}{2} \text{ and } f \geq \frac{1}{2}(R(i^*) - c^*).$$

The first inequality implies $f \leq \frac{\pi(i^*) + i^*}{2}$, while the second inequality leads to $f \geq \frac{\pi(i^*) + i^*}{2}$. These, together, imply $f = \frac{\pi(i^*) + i^*}{2}$ with M1's total net payoff to be given by $\frac{1}{2}(R(i^*) - c^*) - i^*$, which is clearly less than what M1 would obtain by instead investing $\hat{i}$ at $t = 0$. Therefore, the fixed-fee contract with $f = \frac{\pi(i^*) + i^*}{2}$ violates condition (b) — though neither agents would deviate from the contract, but it is irrational to propose such a contract at the very first place.

Following the same argument, cost-sharing linear contract of the type $0.5i - f$ (paying to M1) will not resolve the inefficiency problem. Therefore, the under-investment phenomenon maintains even when the initial investment i is observable and verifiable.

In conclusion, M1's investment $\hat{i}$ is the only stable and rational contract in this example even when the initial investment is verifiable. The first best outcome can be achieved by imposing a big penalty associating with any violation of a contract.⁷ It could be also realized in an economy with bounded rational agents. This is a topic that we do not wish to pursue in this paper.

⁶Note that $\pi(i^*) - f < \frac{1}{2}(R(\hat{i}) - c^*) - \hat{i}$ is not acceptable for M1 since M1 can ensure the final payoff $\frac{1}{2}(R(\hat{i}) - c^*) - \hat{i}$ by not signing the contract.

⁷There is no justification why we should assume parties bind at the contract except by introducing a third party — the court, and by imposing penalties to agents violating the contract. Such modification eventually changes the nature of the game. It changes not only the strategic structures, but also the payoffs of the game.

1.6 Main Contributions and Organization of the Paper

In summary, this paper is to show that both coalition-formation / strategic partnership and agents' attitudes towards uncertainty are relevant in contract formation, and constitute contributing factors to rationalize the existence of incomplete contracts, a common phenomenon in real life situations.

The rest of the paper proceeds as follows: Section 2 provides a set-up of the contractual problem. Strategic interaction among agents in absence of a contract is to be studied in Section 3. Section 4 is to consider a situation in which agents are bounded by complete contracts, while interaction in presence of incomplete contracts is the topic covered in Section 5. A newly proposed notion of equilibrium contract, as solution to extensive games of perfect information, will be proposed, and be formulated in Section 6. This section explores also the conditions for the existence of equilibrium, among other properties of the equilibrium contract. Section 7 contains some concluding remarks. Proofs are summarized in the Appendix.

2 Set-up of the Model

The sequential decision problem is summarized by

$$\Gamma \equiv (N, G; U; \Omega, \Xi, \Lambda), \quad (3)$$

where

- N is a finite set of agents.
- $G \equiv (V, E)$ is a directed graph representing a finite extensive decision tree: at each decision node, or event, $v \in V, E(v) \subseteq E$ is the set of feasible actions at v . Action $e \in E(v)$, leads to a unique new decision node indexed by e itself. Therefore, $E(v)$ is regarded as the set of immediate successor nodes of v . Here, we assume that $E(v) \cap E(v') = \emptyset$, for all $v \neq v'$. The set of all terminal nodes is denoted as Z .
- U represents payoffs assigned to each agent at the terminal nodes Z ; i.e., $U_i : Z \rightarrow \mathcal{R}, i \in N$. It induces a unique preference ordering over the set of paths Ω in the graph for each of the agents. Let $u_i : \Omega \rightarrow \mathcal{R}$ be agent i 's preference over the paths.

- A strategy system is a set-valued strategy profiles. Let Ξ be the set of all strategy systems. Therefore, for each $\sigma \in \Xi$, we write $\sigma = \{E(\sigma, v)\}_{v \in V}$ where $E(\sigma, v) \subseteq E(v)$ for all v . A strategy system is also referred to as a plan.
- Λ is the set of all belief systems. Given $\Delta \in \Lambda$, for all $i, v, v', \Delta_i(v, v') \subseteq M(\Omega)$ represents agent i 's beliefs regarding the possible paths going through v' when the game is at decision node v .

The following additional useful notations need to be introduced: Let $\{P_i\}_{i \in N}$ be a partition of V , where P_i represents the set of player i 's decision nodes. For simplicity, we assume that there are no simultaneous moves; that is, $P_i \cap P_j = \emptyset$ for $i \neq j$. Furthermore, for all $v \in V$, the sequential decision problem started at v is denoted as Γ_v , $P_i(v)$ represents the set of nodes after v that belongs to P_i , and Ω_v represents the paths going through v . $v \mapsto \omega$ stands for “ v is on path ω ”; and $v \nrightarrow \omega$ stands for “ v is off path ω ”.

We assume that agents' preferences over the strategy systems are updated at each decision node. We introduce the notion of *preference system*:

$$\succeq \equiv \left\{ \succeq_{(v, v')}^i : i \in N, v \in P_i, v' \in P_i(v) \right\}, \quad (4)$$

where $\succeq_{(v, v')}^i$ is a binary relationship defined on Ξ_i for $\Gamma_{v'}$, and it represents player i 's preference ordering over all admissible plans for sub-decision problem $\Gamma_{v'}$, when player i is actually at his decision node v . In other words, given his information at v , his preference over different plans for $\Gamma_{v'}$ is summarized by $\succeq_{(v, v')}^i$. A preference system $\succeq$ is *dynamically consistent* if, for all i, σ_i and σ'_i ,

$$\sigma_i \succeq_{(v, v')}^i \sigma'_i \iff \sigma_i \succeq_{(v', v')}^i \sigma'_i, \text{ for all } v \in P_i \text{ and } v' \in P_i(v). \quad (5)$$

Consistent with the multi-prior model of decision making in a static environment described in the previous section, we consider the following specification of preference system for the above sequential decision problems. Given a belief system Δ_i , and other players' strategy system σ_{-i} , player i 's preference $\succeq_{(v, v')}^i$ is represented by a utility function $u_i(v; \sigma_i, v')$, which is defined to be such that, for all $v \in P_i, v' \in P_i(v)$ and $\sigma_i(v) = (E(\sigma_i, v'))_{v' \in P_i(v)} \subseteq \Sigma_i(v)$

$$u_i(v; \sigma_i, v') \equiv \max_{e \in E(\sigma_i, v')} \min_{p \in \Delta_i(\sigma_i, v, e)} \int_{\Omega_e} u_i(\omega) dp(\omega), \forall v' \in P_i(v), \quad (6)$$

where $\min_{p \in \Delta_i(\sigma_i; v, e)} \int_{\Omega_e} u_i(\omega) dp(\omega)$ is the most conservative payoff that player i can get by taking action e at v' based on his beliefs $\Delta_i(\sigma_i; v, e)$. In (6), we assume that player i 's plan of action at his other decision nodes affecting his utility at (v, v') is determined by its beliefs. That is, for each plan σ_i and an immediate act e within the planned acts at v' , player i forms a belief $\Delta_i(\sigma_i; v, e)$ regarding the possible paths to be followed after e . Such a belief is formed by taking into consideration his planned actions in future decision nodes. Therefore, we can say that the right hand side of Eq.(6) is the maximum least well-off that player i can achieve at v' through the plan σ_i . For terminal node $z \in Z$, the utility $u_i(v; \sigma_i, z)$ coincides with $u_i(z)$.

3 A World Without A Contract

In a world without a contract, we assume that all agents behave rationally, and that all agents are rational is common knowledge:

- (a) Nobody pre-commits on any specific sequence of actions.
- (b) No communication is allowed among players during the course of actions.
- (c) It is common knowledge that all players are rational and uncertainty averse.

The question becomes: What are the possible actions to be contacted by the players given (a), (b) and (c)?

Definition 1 *A solution for Γ in the absence of a contract is a strategy system $\sigma^* \in \Xi$ such that:*

- (a) *There exists a belief system $\Delta^* \in \Lambda$ that is **consistent** with σ^* : For all $v \in V$ and $v' \in P(v)$, $\Delta^*(\sigma^*; v, v') = M(\Omega_{v'}^*)$, where $\Omega_{v'}^*$ consists of all paths that are generated by σ^* within the sub-decision problem at v' .*
- (b) $\sigma_i^* = (E(\sigma_i^*, v))_{v \in P_i}$, $\forall i \in N$, where for all $v \in P_i$,

$$E(\sigma_i^*, v) = \arg \max_{e \in E(v)} \min_{\omega \in \Omega_e^*} u_i(\omega). \quad (7)$$

In the above definition, condition (a) is a consistency restriction on players' belief system. Every player knows that all players commit to the strategy system σ^* , but is uncertain (vague) about the realization of any particular path within the set of paths induced by σ^* . Condition (b) is an optimality and rationality restriction on agents' formation of plans: First, every act $e \in E(\sigma_i^*, v)$ in the planned action set at v must be optimal to player i at v in the sense that it maximizes the right hand side of Eq.(7). Second, any act that maximizes i 's utility at v , should be included in his "plan" of actions at that node.

Theorem 1 *Given Γ , there exists a unique non-empty solution σ^* in the absence of a contract. Moreover, the set of paths $\{\Omega_v^*\}_{v \in V}$ induced by σ^* uniquely solves the following recursive system: for $v \in Z, \Omega_v^* = \{v\}$; and for $v \in P_i, i \in N$,*

$$\Omega_v^* = \left\{ (e, \omega) \in \Omega_v : \omega \in \Omega_e^* \text{ and } e \in \arg \max_{e \in E(v)} \min_{\omega \in \Omega_e^*} u_i(\omega) \right\}. \quad (8)$$

Proof. See Appendix 1. ■

Therefore, paths associated with the underlying strategy system σ^* can be computed recursively by backward induction.

3.1 Value of Γ in the Absence of A Contract

We start by asking the following question: how much one is willing to pay in order to play a role as player i in Γ ? The value of Γ , which varies among the roles assigned, depends on how the game is to be played. The latter is, of course, determined by players' preferences and their attitude toward uncertainty in particular.

A *value function* u^* for Γ is a map $u^* : N \times V \rightarrow R$ such that, for all $i \in N$ and $v \in V$, $u_i^*(v)$ is the value of Γ_v to i . We assume the following:

Axiom (U1) For $v \in P_i, u_i^*(v) = \max_{e \in E(v)} u_i^*(e)$.

Axiom (U2) For $v \in P_i, j \neq i, u_j^*(v) = \min_{e \in E^*(v)} u_j^*(e)$, where

$$E^*(v) \equiv \arg \max_{e \in E(v)} u_i^*(e).$$

Axiom (U3) For $z \in Z, u_i^*(z) = U_i(z)$, for all $i \in N$.

In the above, Axiom (U1) says that i 's value at his decision node v is the maximum value among all subgames to which he can lead. According to Axiom (U2), player j 's value $u_j^*(v)$ at i 's node v is the most conservative value for j among those subgames $\{\Gamma_e\}_{e \in E^*(v)}$ that are likely to be led by i from v . Axiom (U3) is a “boundary condition” for the value function u^* at terminal nodes in Z .

Theorem 2 *Given Γ , there exists a unique value function $u^* : N \times V \rightarrow R$ that satisfies Axioms (U1), (U2) and (U3). Moreover, the strategy system $(E^*(v))_{v \in P}$ defined in Axiom (U2) coincides with σ^* ; and in particular, $u_i(\sigma^*, v) = u_i^*(v)$ for all $v \in P_i$ and for all i .*

Proof. See Appendix 2. ■

Therefore, we find a unique value for every subgame to each player. The value of Γ in the absence of a contract is achieved at the σ^* -solution.

4 Solution in Presence of A Complete Contract

This section considers an economy in which agents can sign a complete contract. A contract σ is said to be *complete* if, for all i , $E(\sigma_i, v)$ is singleton at $v \in P_i$.

A complete contract σ is associated with a unique path in Ω denoted as ω_σ . This path is referred to as the contractual path. A complete contract specifies also a unique (truncated) contractual path for each subgame, even for those off the contractual path ω_σ . This raises the following questions: how can one expect the players to honor a contract when they find themselves off the contractual path ω_σ ? If not, what would guide their actions along paths that are off the contractual paths? Therefore, we make the following assumptions:

- A complete contract σ is known before actions taking place.
- Communication is prohibited during the play of the game.
- Agents have the option to act according to the contract or not to, and they will follow the contractual path as long as it is rational for them to do so, and as long as they find themselves on the contractual path.
- It is common knowledge at all decision nodes that agents are uncertainty averse and rational.

The third assumption says that when an agent faces a set of actions that are equally the best, he will choose the one specified in the contract if that act is in the optimal set. Otherwise, when agents find themselves off the contractual path, the last assumption implies that what guides their action-takings is the common knowledge of uncertainty averse rationality, not what is suggested by the (violated) contract!

Definition 2 *Given a complete contract $\sigma \equiv (E(\sigma_i, v))_{\substack{v \in V \\ i \in N}} \in \Xi$, the solution to Γ is a strategy system $\hat{\sigma} \in \Xi$ such that:*

- (a) There exists a belief system $\hat{\Delta} \in \Lambda$ that is consistent with $\hat{\sigma} \in \Xi$: For all $v \in V$, and $v' \in P(v)$, $\hat{\Delta}(\hat{\sigma}; v, v') = M(\hat{\Omega}_{v'})$, where $\hat{\Omega}_{v'}$ consists of all paths generated by $\hat{\sigma}$ within the subgame $\Gamma_{v'}$.
- (b) $\hat{\sigma}_i = (E(\hat{\sigma}_i, v))_{v \in P_i}$, $\forall v \in P_i, i \in N$, be such that
 - (1) for $v \mapsto \omega_\sigma$, $E(\hat{\sigma}_i, v) = \arg \max_{e \in E(v)} \min_{\omega \in \hat{\Omega}_e} u_i(\omega)$;
 - (2) for $v \mapsto \omega_\sigma$, $E(\hat{\sigma}_i, v) = E(\sigma_i, v)$ if $E(\sigma_i, v) \in E^*(\hat{\sigma}_i, v)$; and $E(\hat{\sigma}_i, v) = E^*(\hat{\sigma}_i, v)$ if $E(\sigma_i, v) \notin E^*(\hat{\sigma}_i, v)$, where

$$E^*(\hat{\sigma}_i, v) = \arg \max_{e \in E(v)} \min_{\omega \in \hat{\Omega}_e} u_i(\omega). \quad (9)$$

- (c) For all $z \in Z$, $\hat{\sigma}_i(z) = z$.

Theorem 3 *Given a complete contract $\sigma \in \Xi$, the solution $\hat{\sigma}$ of Γ is non-empty and unique. In addition, $\hat{\sigma}$ has the following properties:*

- (i) Paths $\{\hat{\Omega}_v\}_{v \in V}$ induced by $\hat{\sigma}$ solves uniquely the following recursive system: for $v \in Z$, $\hat{\Omega}_v = v$; for $v \in P_i, i \in N$, if $v \mapsto \omega_\sigma$, and $E(\sigma_i, v) \in E^*(\hat{\sigma}_i, v)$, then $\hat{\Omega}_v = \hat{\Omega}_e, e = E(\sigma_i, v)$; otherwise, if either $v \mapsto \omega_\sigma$ and $E(\sigma_i, v) \notin E^*(\hat{\sigma}_i, v)$, or $v \mapsto \omega$, we have $\hat{\Omega}_v = \Omega_v^*$;
- (ii) If $\omega_\sigma \in \Omega^*$, then for all $v \mapsto \omega_\sigma$, $\hat{\Omega}_v = \{\omega_\sigma\}$.

Proof. See Appendix 3. ■

Property (ii) says that if the contractual path belongs to the set of paths induced by the solution of Γ in the absence of a contract, then such a path will be carried out. In general, even if a contract is complete, the underlying solution may involve set-valued acts.

4.1 The Set of Stable Complete Contracts

Here we focus on those stable complete contracts — being *stable* in the sense that once proposed, it will be carried out in the solution; i.e., $\omega_\sigma \in \widehat{\Omega}$. For any given $v \in V$, let $\Omega_v^{**} \subseteq \Omega_v$ be the set of all stable paths in Ω_v . That is,

$$\Omega_v^{**} = \left\{ \omega_v \in \Omega_v : \text{for all } v' \mapsto \omega_v, \omega_{v'} \in \widehat{\Omega}_{v'} \right\}. \quad (10)$$

Following from theorem 3, we see that the set Ω_v^* of all paths generated by the σ^* -solution belongs to Ω_v^{**} . That is:

Corollary 1 *The set of Ω^{**} -stable contractual paths is non-empty, and contains the set of σ^* -solution paths as a subset.*

The above belonging relationships can hold strictly. Consider the game described in Figure 1.

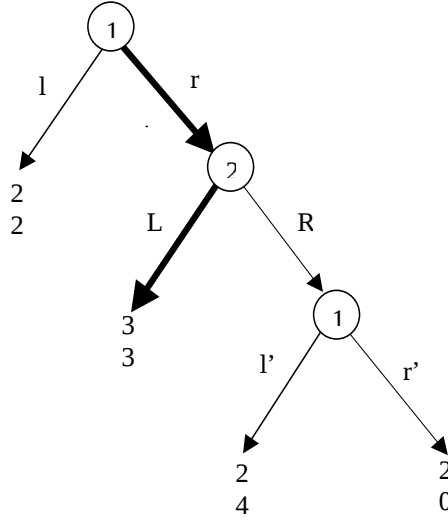


Figure 1

This game has a unique σ^* -path $\Omega^* = \{(r, L)\}$ with two stable contractual paths $\Omega^{**} = \{(r, L); (r, R, l')\}$.

4.2 Value of Γ in the Presence of A Complete Contract σ

The value of Γ in the presence of a complete contract σ , which is also referred to as the value of contract σ , is a price vector that agents are willing to pay

in order to play the game Γ and each of the subgames in the presence of a contract σ . Similar to the economy in the absence of a contract, the value function $u^*(\sigma; \cdot) : N \times V \rightarrow \mathcal{R}$ is assumed to satisfy the following axioms:

Axiom (UC1) For $v \in P_i$, $u_i^*(\sigma; v) = \max_{e \in E(v)} u_i^*(\sigma; e)$.

Axiom (UC2) For $v \in P_i$, $j \neq i$, $u_j^*(\sigma; v) = \min_{e \in E^*(\sigma; v)} u_j^*(\sigma; e)$, where $E^*(\sigma; v) = \{E(\sigma, v)\}$ if $v \mapsto \omega_\sigma$ and if $E(\sigma, v) \in \arg \max_{e \in E(v)} u_i^*(\sigma; e)$; and $E^*(\sigma; v) = \arg \max_{e \in E(v)} u_i^*(\sigma; e)$, otherwise.

Axiom (UC3) For $z \in Z$, $u_i^*(\sigma; z) = U_i(z)$, for all $i \in N$.

In the above, Axiom (UC2) requires some explanation. Player j 's value $u_j^*(\sigma; v)$ at i 's node v is the most conservative value for j among the values for those subgames $\{\Gamma_e\}_{e \in E^*(\sigma; v)}$ that are likely to be led by i from v , keeping in mind that agent i is not to deviate from the contractual act at v unless it is either off the contractual path, or on the contractual path but irrational for i to follow the contractual act at v .

Similar to the economy in the absence of a contract, we can show that:

Theorem 4 *The value of contract σ satisfying Axioms (UC1)—(UC3) is unique, and is given by agents' uncertainty averse utilities at $\hat{\sigma}$.*

Proof. See Appendix 4. ■

Therefore, each contract corresponds to a unique value for each player of the game. The question is: does the existence of contracts benefit all parties in the sense that it leads to a higher value for each agent than what would be achieved in a world without a contract?

Examples can be constructed in showing that the answer to the above question is not affirmative! In a free world, those who become worse off in the presence of a contract can indeed refuse to sign the contract in the first place. This, of course, may results in incomplete contracts where some players' strategies are not specified. Conversely, a group of players may want to form a coalition in forming a contract among themselves aiming to induce a higher value for each member of the coalition. A contract like this will leave the actions of those members outside the coalition unspecified. In either situation, we have to deal with incomplete contracts.

5 Solution in Presence of A (Possibly) Incomplete Contract

This section considers the situation when agents can write an incomplete contract before actions take place. Similar to the case with a complete contract, we assume the following:

- A contract σ is publicly announced.
- Communications are prohibited during the play of the game.
- Agents have the option to follow the announced contract, or not to.
- Agents are not willing to contradict to the announced contract unless it is irrational for them to follow the contract.
- It is common knowledge that all agents are uncertainty averse and rational.

Definition 3 *Given a contract $\sigma \equiv (E(\sigma_i, v))_{\substack{v \in V \\ i \in N}} \in \Xi$, the solution to Γ is a strategy system $\hat{\sigma} \in \Xi$ such that:*

- (a) There exists a belief system $\hat{\Delta} \in \Lambda$ that is consistent with $\hat{\sigma} \in \Xi$: For all $v \in V$, and $v' \in P(v)$, $\hat{\Delta}(\hat{\sigma}; v, v') = M(\hat{\Omega}_{v'})$, where $\hat{\Omega}_{v'}$ consists of all paths generated by $\hat{\sigma}$ within the subgame $\Gamma_{v'}$.
- (b) For all i and $v \in P_i$, $\hat{\sigma}_i = (E(\hat{\sigma}_i, v))_{v \in P_i}$, be such that
 - (1) $E(\hat{\sigma}_i, v) = E(\sigma_i^*, v)$ if $v \rightsquigarrow \omega$ for all $\omega \in \Omega_\sigma(v^*)$.
 - (2) For $v \mapsto \omega \in \Omega_\sigma(v^*)$, $E(\hat{\sigma}_i, v) = E(\sigma_i, v) \cap E^*(\hat{\sigma}_i, v)$ if it is non-empty; otherwise, $E(\hat{\sigma}_i, v) = E^*(\hat{\sigma}_i, v)$, where

$$E^*(\hat{\sigma}_i, v) = \arg \max_{e \in E(v)} \min_{\omega \in \Omega_e} u_i(\omega). \quad (11)$$

- (3) For all $v \in Z$, $\hat{\sigma}_i(v) = v$.

We have:

Theorem 5 *Given Γ and a contract $\sigma \in \Xi$, the solution $\hat{\sigma}$ is non-empty and unique. In addition, $\hat{\sigma}$ has the following properties:*

- (i) Paths $\{\widehat{\Omega}_v\}_{v \in V}$ induced by $\widehat{\sigma}$ solves uniquely the following recursive system: For $v \in Z$, $\widehat{\Omega}_v = v$. For $v \in P_i, i \in N$, if $v \mapsto \omega \in \Omega_\sigma(v^*)$ and if $E(\sigma_i, v) \cap E^*(\widehat{\sigma}_i, v) \neq \emptyset$,

$$\widehat{\Omega}_v = \left\{ (e, \omega) : \omega \in \widehat{\Omega}_e, e \in E(\sigma_i, v) \cap E^*(\widehat{\sigma}_i, v) \right\}; \quad (12)$$

otherwise $\widehat{\Omega}_v = \Omega_v^*$.

- (ii) For all $v \in V$, and $v \mapsto \omega \in \Omega_\sigma(v^*)$, if $\sigma(v) \subseteq \sigma^*(v)$, then $\widehat{\sigma}(v) \subseteq \sigma(v)$.

Proof. See Appendix 5. ■

5.1 Value of Γ in the Presence of Contract σ

Similar to previous sections, the value function $u^*(\sigma; \cdot) : N \times V \rightarrow \mathcal{R}$ is assumed to satisfy the following axioms:

Axiom (UI1) For $v \in P_i, u_i^*(\sigma; v) = \max_{e \in E(v)} u_i^*(\sigma; e)$.

Axiom (UI2) For $v \in P_i, j \neq i, u_j^*(\sigma; v) = \min_{e \in E^*(\sigma; v)} u_j^*(\sigma; e)$, where $E^*(\sigma; v) = E(\sigma, v) \cap \arg \max_{e \in E(v)} u_i^*(\sigma; e)$ if $v \mapsto \omega \in \Omega_\sigma$ and if $E(\sigma, v) \cap \arg \max_{e \in E(v)} u_i^*(\sigma; e) \neq \emptyset$; and $E^*(\sigma; v) = \arg \max_{e \in E(v)} u_i^*(\sigma; e)$, otherwise.

Axiom (UI3) For $z \in Z, u_i^*(\sigma; z) = U_i(z)$, for all $i \in N$.

We can show that,

Theorem 6 *The value of contract σ satisfying Axioms (UI1)–(UI3) is unique, and is given by agents' uncertainty averse utilities at $\widehat{\sigma}$.*

Proof. See Appendix 6. ■

We see that every contract has a unique value for each player of the game. Does the existence of a contract make everybody better off? The answer is, of course, not always true. Will the contract be carried out? If not, why shall they sign the contract at the first place? After all, we want to understand how rational agents form their contracts.

6 A Theory of Equilibrium Contract

In this section, we tackle the questions raised at the end of the previous section. We start by introducing the notions of *core* and Ξ -*equilibrium contract*.

Definition 4 *A contract σ is said to be in the core if a coalition $S \subseteq N$ does not exist with strategy system $\sigma'_S \neq \sigma_S$ and $\sigma'_{-S} = \sigma_{-S}$ such that⁸*

$$u_i^*(\sigma'; v^*) > u_i^*(\sigma; v^*) \text{ for all } i \in S. \quad (13)$$

Here, $u^*(\sigma; \cdot)$ is the value function of contract σ defined in the previous section, which is given by the uncertainty averse utility at $\hat{\sigma}$, the solution of Γ in the presence of contract σ . Here, we do not require σ' to be stable, or to be self-enforcing in the sense of BPW (1987). Nevertheless, the stability restriction constitutes part of the restriction in defining an equilibrium contract. That is, .

Definition 5 *Given Γ , σ is called an Ξ -equilibrium if it satisfies the following two conditions:*

- (a) σ is stable; that is, $\hat{\sigma} = \sigma$.
- (b) σ is in the core.

To illustrate the newly proposed solution, consider again the game described in Figure 1. This game has two stable contracts: (a) player 1 chooses r and “either l' or r' ”, while player 2 takes “ L ”; (b) player 1 takes “ r ” and “ l' ”, while player 2 takes “ R ”. Contract (a) is in the core because nobody can benefit for sure by proposing other plans. System (b) is not in the core, because any other plan proposed by player 1, such as “ l' ” or “either l' or r' ”, will result in a better final outcome (r, L) for sure. Therefore, this game has a unique Ξ -equilibrium path (r, L) .

Let $\hat{\Xi}$ be the set of all *stable* contracts. The proposition below shows that the set of stable contracts is non-empty.

Theorem 7 *For all $\sigma \in \Xi$, the solution $\hat{\sigma}$ under contract σ is stable, i.e., $\hat{\sigma} \in \hat{\Xi}$.*

⁸We may also define the core by imposing instead the weak inequalities for all coalition members, and with the strict inequalities for some coalition members.

Proof. See Appendix 7. ■

It can be further shown that the set of stable contracts refine the subgame perfect equilibrium in the sense that, each stable path must be supported by a subgame perfect equilibrium path (see Ma 2000, Theorem 6-(d)). Therefore, the Ξ -equilibrium constitutes also a refinement of subgame perfect equilibrium if it exists. The following provides a sufficient condition for the existence of Ξ -equilibrium.

Theorem 8 *Given Γ , if a stable plan $\sigma \in \Xi$ be such that there does not exist a coalition S and another stable plan σ' such that⁹*

$$u_S^*(\sigma'; v^*) \gg u_S^*(\sigma; v^*), \quad (14)$$

then σ is an Ξ -equilibrium.

Proof. See Appendix 8. ■

Remark 1 *Condition (14) is not necessary for the existence of Ξ -equilibrium. Consider the game described in Figure 2.*

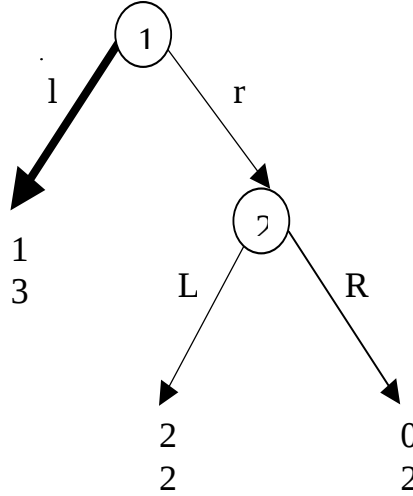


Figure 2

The unique Ξ -equilibrium path $\{l\}$ violates obviously condition (14).

⁹For all $x, y \in \mathcal{R}^n$, $x \gg y$ if and only if $x_i > y_i$ for all i .

Remark 2 Ξ -equilibrium may or may not contain the σ^* -solution. In Figure 2, the unique Ξ -equilibrium is given by the σ^* -solution $(l, \{L, R\})$. However, if the payoff for player 2 at l is 1 instead of 3, then the game has a unique Ξ -equilibrium (r, L) , while the σ^* -solution is still given by $(l, \{L, R\})$.

Remark 3 Consider the game described in Figure 3.

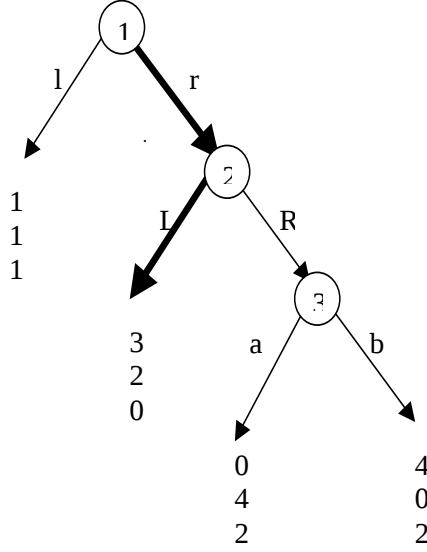


Figure 3

It has no strong Nash equilibrium and no coalition-proof Nash equilibrium. Nevertheless, it has a unique Ξ -equilibrium.

7 Concluding Remarks

This paper provides a theory of incomplete contracts within the context of multi-agent sequential decision making with perfect information. A stable contract is not necessarily in the equilibrium set since rational agents may have no incentive to sign such a contract, keeping in mind that once signed, it will be followed. An equilibrium contract is not necessarily complete, nor is necessarily Pareto optimal. The equilibrium contract could be strictly Pareto dominated even when it is the unique Ξ -equilibrium. Finally, while it refines the Nash equilibrium of Nash (1953) and the subgame perfect equilibrium of Selten (1975), the proposed equilibrium contract differs also from the other well-known refinements in the literature such as the strong Nash

equilibrium of Aumann (1959) and the coalition-proof Nash equilibrium of BPW (1987).

A 1. Proof of Theorem 1

First, by backward induction, equation (8) has a unique solution $\{\Omega_v^*\}_{v \in V}$ since at each decision node only finite acts exist. For all $v \notin Z$, let

$$E^*(v) = \{e \in E(v) : \text{there exists } \omega \in \Omega_e^* \text{ such that } (e, \omega) \in \Omega_v^*\}. \quad (15)$$

Let $\sigma^* = (E^*(v))_{v \in V}$, and for all $v \in V$, and $v' \in P(v)$, set $\Delta^*(\sigma^*; v, v') = M(\Omega_{v'}^*)$. The pair (σ^*, Δ^*) satisfies obviously the conditions of Definition 1.

Second, let (σ', Δ') satisfy the two conditions in Definition 1 with $\{\Omega_v'\}_{v \in V}$ generated by σ' . We have, for all $v \notin Z$, $\Omega_v' = \{(e, \omega) : e \in E(\sigma', v) \text{ and } \omega \in \Omega_e'\}$. The paths $\{\Omega_v'\}_{v \in V}$ generated by σ' solves also the equation (8); hence coincides with the unique solution $\{\Omega_v^*\}_{v \in V}$. Therefore, σ' must coincide with σ^* because they induce to the same paths of play. This concludes the proof of Theorem 1.

A 2. Proof of Theorem 2

The existence and uniqueness of the value function follows by the three axioms: The value function at the terminal nodes coincides with $u : Z \rightarrow \mathcal{R}^n$, is uniquely well-defined. Assuming that the value function is uniquely defined for all subgames with length no longer than $l \geq 1$, let v be a node to be such that the subgame starting from v has a length of $l + 1$. By Axioms U1 and U2, the value function at v is also well-defined.

The second part of the theorem follows also by induction. The statement holds for all subgames of length 1. Suppose it holds for all subgames whose length are no longer than $l (\geq 1)$. Consider a subgame Γ_v has a length of $l + 1$, and is rooted from $v \in V$. Without loss of generality, we assume $v \in P_i$. By assumption, for all $e \in E(v)$, in subgame Γ_e we have $u^*(i, e) = \min_{\omega \in \Omega^*(e)} u_i(\omega)$. Therefore, $e \in E^*(v)$ if and only if $e \in \arg \max_{e \in E(v)} \min_{\omega \in \Omega^*(e)} u_i(\omega) \equiv E(\sigma^*, v)$. We conclude that $(E^*(v'))_{v' \in P(v)}$ coincides with $\sigma^*(v) \equiv (E(\sigma^*, v'))_{v' \in P(v)}$. Moreover, by Axiom (U1), for player i , we have

$$u^*(i, v) \equiv \max_{e \in E(v)} u^*(i, e) = \max_{e \in E(v)} \min_{\omega \in \Omega^*(e)} u_i(\omega) = \min_{\omega \in \Omega^*(v)} u_i(\omega),$$

for all $e \in E(\sigma^*, v)$, or equivalently, $u^*(i, v) = \min_{\omega \in \Omega^*(v)} u_i(\omega)$; similarly, for $j \neq i$, by Axiom (U2), we have

$$u^*(j, v) \equiv \min_{e \in E^*(v)} u^*(j, e) = \min_{e \in E(\sigma^*, v)} \min_{\omega \in \Omega^*(e)} u_j(\omega) = \min_{\omega \in \Omega^*(v)} u_j(\omega).$$

This ends the proof of theorem 2.

A 3. Proof of Theorem 3

The first statement of the theorem and property (i), similar to the proof of Theorem 1, follow by the definition of $\widehat{\sigma}$ -equilibrium and the backward induction argument.

We proceed the proof of statement (ii) by induction. The statement holds obviously for $l = 1$. Assume that (ii) holds for all subgames that have a length less than or equal to l , and suppose, on the contrary, that there exists an ω and $v \in P_i$ such that $\omega_v \in \Omega_v^*$, but $\omega_v \notin \widehat{\Omega}_v$, and that the length of ω_v is $l + 1$. Note that $\omega_{v'} \in \widehat{\Omega}_{v'}, \forall v \neq v' \mapsto \omega_v$. Therefore, the condition $\omega_v \notin \widehat{\Omega}_v$ implies that there exists an $e \in E(v), e \neq \omega(v)$ such that

$$\begin{aligned} u_i(\omega_v) &< \min_{\omega' \in \widehat{\Omega}_v} u_i(\omega') \\ &= \min_{\omega' \in \Omega_e^*} u_i(\omega') \\ &\leq \max_{e \in E(v)} \min_{\omega' \in \Omega_e^*} u_i(\omega'), \end{aligned}$$

where the equality follows from property (i) that $\widehat{\Omega}_e = \Omega_e^*$ since, by assumption, $e \mapsto \omega$. This inequality contradicts to the assumption that $\omega_v \in \Omega_v^*$ by theorem 1. This ends the proof of Theorem 3.

A 4. Proof of Theorem 4

The existence and uniqueness of the value function follows by the three axioms: The value function at the terminal nodes coincides with $u : Z \rightarrow \mathcal{R}^n$, is thus uniquely well-defined. Assuming that the value function is uniquely defined for all subgames with length no longer than $l \geq 1$, let v be a node to be such that the subgame starting from v has a length of $l + 1$. By Axioms UC1 and UC2, the value function at v is also well-defined.

The second part of the theorem follows also by induction. The statement holds for all subgames of length 1. Suppose it holds true for all subgames whose length are no longer than $l (\geq 1)$. Consider a subgame

Γ_v has a length of $l + 1$, which is rooted from $v \in V$. Without loss of generality, we assume $v \in P_i$. By assumption, for all $e \in E(v)$, in subgame Γ_e we have $u^*(i, e) = \min_{\omega \in \hat{\Omega}(e)} u_i(\omega)$. Therefore, $e \in E^*(v)$ if and only if $e \in \arg \max_{e \in E(v)} \min_{\omega \in \Omega^*(e)} u_i(\omega) \equiv E(\sigma^*, v)$. We conclude that $(E^*(v'))_{v' \in P(v)}$ coincides with $\sigma^*(v) \equiv (E(\sigma^*, v'))_{v' \in P(v)}$. Moreover, by Axiom (UC1), for player i , we have

$$u^*(i, v) \equiv \max_{e \in E(v)} u^*(i, e) = \max_{e \in E(v)} \min_{\omega \in \Omega^*(e)} u_i(\omega) = \min_{\omega \in \Omega^*(v)} u_i(\omega),$$

for all $e \in E(\sigma^*, v)$, or equivalently, $u^*(i, v) = \min_{\omega \in \Omega^*(v)} u_i(\omega)$; similarly, for $j \neq i$, by Axiom (UC2), we have

$$u^*(j, v) \equiv \min_{e \in E^*(v)} u^*(j, e) = \min_{e \in E(\sigma^*, v)} \min_{\omega \in \Omega^*(e)} u_j(\omega) = \min_{\omega \in \Omega^*(v)} u_j(\omega).$$

This ends the proof of theorem 4.

A 5. Proof of Theorem 5

We proceed to prove first, by induction, the first statement and part (i) of the second statement in the theorem. The $\hat{\sigma}$ -solution is obviously well-defined for subgames of length 1. Suppose $\hat{\sigma}$ -solution is well-defined for all subgames with a length no longer than l . Let Γ_v have a length of $l + 1$. Two cases are distinguished:

Case 1: v is off the contractual paths. The solution $\hat{\sigma}$ for the subgame Γ_v is unique and coincides with σ^* for Γ_v following from Definition 3, (b)-(1) and Definition 1, and noting that all nodes in Γ_v will be also off the contractual paths. In particular, we have $\hat{\Omega}_v = \Omega_v^*$.

Case 2: v is on the contractual paths. Without loss of generality, let $v \in P_i$. Solution $\hat{\sigma}$ for the subgame Γ_v is unique, and is determined by condition (b)-(2), Definition 3. Two sub-cases are distinguished:

a) $E(\sigma_i, v) \cap E^*(\hat{\sigma}_i, v)$ is empty; i.e., none of the contractual acts at v is locally optimal. We have $E(\hat{\sigma}_i, v) = E^*(\hat{\sigma}_i, v)$ determined by equation (11). Since all nodes in $E^*(\hat{\sigma}_i, v)$ are necessarily off the contractual paths, these result in σ^* -solution for all subgames $\Gamma_e, e \in E^*(\hat{\sigma}_i, v)$, following from Case 1. Therefore, from equation (11), we have $E^*(\hat{\sigma}_i, v) = E(\sigma_i^*, v)$. In particular, we have again $\hat{\Omega}_v = \Omega_v^*$.

b) $E(\hat{\sigma}_i, v) = E(\sigma_i, v) \cap E^*(\hat{\sigma}_i, v) \neq \emptyset$. In that case, the solution path is obviously given by the right hand side of equation (12).

To prove statement (ii) of the theorem, notice that the contractual paths form a subset of the σ^* -solution paths. For v off the contractual paths, the solution for the subgame Γ_v coincides with the σ^* -solution following the same logic used in Case 1 above. For v on the contractual paths, we always have $E(\hat{\sigma}_i, v) = E(\sigma_i, v) \cap E^*(\hat{\sigma}_i, v)$, which is non-empty.¹⁰ In particular, we have $E(\hat{\sigma}_i, v) \subseteq E(\sigma_i, v) \subseteq E(\sigma_i^*, v)$. This holds for all i and $v \in P_i$. Therefore, we conclude that $\hat{\sigma}(v) \subseteq \sigma(v) \subseteq \sigma^*(v)$.

A 6. Proof of Theorem 6

The proof proceeds in two steps. First, following backward induction, we show that the value function is uniquely defined. Second, we need to verify that the uncertainty averse utility at $\hat{\sigma}$ satisfies all three axioms; therefore must coincide with the value function. The details of the proof are similar to the proofs of Theorems 2 and 4 above, and are therefore omitted. This ends the proof of Theorem 6.

A 7. Proof of Theorem 7

Let $\hat{\sigma}$ be the solution of Γ in the presence of contract σ . We need to show that, $\hat{\sigma}' = \hat{\sigma}$ for the contract $\sigma' = \hat{\sigma}$. We prove this by induction.

The statement is true obviously for subgames with length 1. Suppose it is true for all subgames with a length no longer than $l \geq 1$. Consider a subgame starting at v with length $l + 1$. Without loss of generality, let $v \in P_i$. For v that is off the contractual paths $\hat{\Omega}$, we have, by theorem 5(i)&(ii), $E(\hat{\sigma}'_i, v) = E(\sigma_i^*, v) = E(\hat{\sigma}_i, v)$. For v that is on the contractual paths $\hat{\Omega}$, we have, from equation (11),

$$E^*(\hat{\sigma}'_i, v) = \arg \max_{e \in E(v)} \min_{\omega \in \hat{\Omega}'_e} u_i(\omega) = \arg \max_{e \in E(v)} \min_{\omega \in \hat{\Omega}_e} u_i(\omega) = E^*(\hat{\sigma}_i, v)$$

since, by assumption, $\hat{\Omega}'_e = \hat{\Omega}_e$. We have, $E(\hat{\sigma}'_i, v) = E(\hat{\sigma}_i, v) \cap E^*(\hat{\sigma}'_i, v) = E(\hat{\sigma}_i, v) \cap E^*(\hat{\sigma}_i, v) = E(\hat{\sigma}_i, v)$ by noting that $E(\hat{\sigma}_i, v) \subseteq E^*(\hat{\sigma}_i, v)$. Therefore, $E(\hat{\sigma}'_i, v) = E(\hat{\sigma}_i, v)$. This implies $\hat{\sigma}'(v) = \hat{\sigma}(v)$. This ends the proof of Theorem 7.

¹⁰Suppose, to the contrary, that $E(\sigma_i, v) \cap E^*(\hat{\sigma}_i, v) = \emptyset$. Following the same logic as case a) above, we have $E^*(\hat{\sigma}_i, v) = E(\sigma_i^*, v)$. This contradicts to $E(\sigma_i, v) \cap E^*(\hat{\sigma}_i, v) = \emptyset$ since, by assumption, $E(\sigma_i, v) \subseteq E(\sigma_i^*, v)$.

A 8. Proof of Theorem 8

Suppose, on the contrary, that σ satisfies the stated condition, but σ is not an Ξ -equilibrium. The latter implies, by Definitions 4 & 5, that there exists a coalition S with a new plan $\sigma'_S \neq \sigma_S$ such that for the new contract $\sigma' = (\sigma'_S, \sigma_{-S})$, we have,

$$u_S^*(\sigma'; v^*) \gg u_S^*(\sigma; v^*).$$

By theorem 7, $\widehat{\sigma}'$ is stable. By theorem 6, we have $u_S^*(\widehat{\sigma}'; v^*) = u_S^*(\sigma'; v^*) \gg u_S^*(\sigma; v^*)$. This is a contradiction to the assumption.

This ends the proof of theorem 8.

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